ORIGINAL ARTICLE

Tidal torques: a critical review of some techniques

Michael Efroimsky · James G. Williams

Abstract   We review some techniques employed in the studies of torques due to bodily tides, and explain why the MacDonald formula for the tidal torque is valid only in the zeroth order of the eccentricity divided by the quality factor, while its time-average is valid in the first order. As a result, the formula cannot be used for analysis in higher orders of e/Q. This necessitates some corrections in the current theory of tidal despinning and libration damping (though the qualitative conclusions of that theory may largely remain correct). We demonstrate that in the case when the inclinations are small and the phase lags of the tidal harmonics are proportional to the frequency, the Darwin-Kaula expansion is equivalent to a corrected version of the MacDonald method. The latter method rests on the assumption of existence of one total double bulge. The necessary correction to MacDonald's approach would be to assert (following Singer, Geophys. J. R. Astron. Soc., 15: 205–226, 1968) that the phase lag of this integral bulge is not constant, but is proportional to the instantaneous synodal frequency (which is twice the difference between the evolution rates of the true anomaly and the sidereal angle). This equivalence of two descriptions becomes violated by a nonlinear dependence of the phase lag upon the tidal frequency. It remains unclear whether it is violated at higher inclinations. Another goal of our paper is to compare two derivations of a popular formula for the tidal despinning rate, and emphasise that both are strongly limited to the case of a vanishing inclination and a certain (sadly, unrealistic) law of frequency-dependence of the quality factor Q—the law that follows from the phase lag being proportional to frequency. One of the said derivations is based on the MacDonald torque, the other on the Darwin torque. Fortunately, the second approach is general enough to accommodate both a finite inclination and the actual rheology. We also address the rheological models with the Q factor scaling as the tidal frequency to a positive fractional power, and disprove the popular belief that these models introduce discontinuities into the equations and thus are unrealistic

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Report Documentation Page			Form Approved OMB No. 0704-0188

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1. REPORT DATE **2009**	2. REPORT TYPE		3. DATES COVERED **00-00-2009 to 00-00-2009**
4. TITLE AND SUBTITLE **Tidal Torques: A Critical Review of Some Techniques**			5a. CONTRACT NUMBER
			5b. GRANT NUMBER
			5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)			5d. PROJECT NUMBER
			5e. TASK NUMBER
			5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **US Naval Observatory,Washington,DC,20392**			8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)			10. SPONSOR/MONITOR'S ACRONYM(S)
			11. SPONSOR/MONITOR'S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release; distribution unlimited**			
13. SUPPLEMENTARY NOTES			
14. ABSTRACT			
15. SUBJECT TERMS			

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**33**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

at low frequencies. Although such models indeed make the conventional expressions for the torque diverge at vanishing frequencies, the emerging infinities reveal not the impossible nature of one or another rheology, but a subtle flaw in the underlying mathematical model of friction. Flawed is the common misassumption that damping merely provides phase lags to the terms of the Fourier series for the tidal potential. A careful hydrodynamical treatment by Sir George Darwin (1879), with viscosity explicitly included, had demonstrated that the magnitudes of the terms, too, get changed—a fine detail later neglected as "irrelevant". Reinstating of this detail tames the fake infinities and rehabilitates the "impossible" scaling law (which happens to be the actual law the terrestrial planets obey at low frequencies). Finally, we explore the limitations of the popular formula interconnecting the quality factor and the phase lag. It turns out that, for low values of Q, the quality factor is no longer equal to the cotangent of the lag.

1 Prologue

> *When it shall be found that much is omitted,*
> *let it not be forgotten that much likewise is performed*
> *Samuel Johnson, 1755*

In his short work *"Untersuchung der Frage …,"* known among the historians also as the *"Spin-Cycle essay,"* Immanuel Kant (1754) stated that the Moon not only pulls the Earth, but also exerts a retarding torque upon its surface; this torque slows down the Earth's rotation and lets go only when terrestrial days become as long as lunar months. Although Kant had in mind only the ocean tides, not the bodily ones, we may say that, qualitatively, he predicted the celebrated 1:1 spin–orbit resonance, the *pas de deux* wherein Pluto and Charon are locked.

For the first time, the idea of tidal action not being confined only to the fluid portion of the planet but affecting also the solid, so as to induce a state of varying strain, was put forward by John Herschel (son of astronomer William Herschel), as a minor aside in a paper devoted to volcanism and earthquakes (Herschel 1863). The earliest mathematical description of land tides in their dynamics was offered by George Darwin (son of naturalist Charles Darwin and great-grandson of poet and philosopher Erasmus Darwin).

Following his predecessors Roche (1849) and Thomson (1863), who had calculated the figure of a static tide, Darwin (1879) assumed the Earth homogeneous and consisting of an incompressible fluid. To account for dynamics, he also assumed that the viscosity was the sole source of the tidal friction. Relying on this model, Darwin (1880, 1908) derived a tide-generated disturbing potential expanded into a Fourier series. Substitution thereof into the Lagrange-type planetary equations led him to expressions for the time derivatives of the orbital elements via partial derivatives of the disturbing potential with respect to the elements.

An impressive generalisation of Darwin's work by Kaula (1964), and the subsequent flow of new concepts and applications (MacDonald 1964; Goldreich 1966a,b; Goldreich and Peale 1996; Singer 1968; Mignard 1979, 1980; Touma and Wisdom 1994; Neron de Surgy and Laskar 1997; Krasinsky 2002, 2006; Getino et al. 2003; Ferraz-Mello et al. 2008; Efroimsky 2008) made bodily tides a rapidly developing area of the planetary astronomy. The vast and growing volume of the relevant material leaves one no chance to glean it all in

one review. Therefore, we shall concentrate on one special aspect of this research, the tidal torques emerging from the bodily tides. Moreover, we shall dwell solely on the techniques, not applications.

Although our review will at times be very critical, it should from the beginning be agreed that our criticisms are intended in the spirit of the above quotation from Samuel Johnson.

Along with reviewing the current state of the field, we shall provide some new results of our own. Specifically, we shall address the rheological models with the Q factor scaling as the tidal frequency to a positive fractional exponential. We shall demonstrate that, contrary to the common opinion, such rheologies do *not* cause infinities in the expression for the torque. We shall also derive an expression for the tidal torque decelerating a terrestrial planet obeying such a rheology. (That the realistic terrestrial bodies indeed obey this class of rheologies has been explained in Efroimsky and Lainey (2007).)

2 Trivia

In this section, we shall briefly recall how a satellite-generated potential in a point on or inside the planet is expressed through the latitude, longitude, and the radial distance of the point.

Let us begin from the first principles. The dynamics of point masses m_i located at inertial-frame-related positions $\vec{\rho}_i$,

$$m_i \ddot{\vec{\rho}}_i = m_i \sum_{j \neq i} G m_j \frac{\vec{r}_{ij}}{r_{ij}^3}, \quad \vec{r}_{ij} \equiv \vec{\rho}_j - \vec{\rho}_i, \ i, j = 1, \ldots, N, \tag{1}$$

may be conveniently reformulated in terms of the relative-to-the-primary locations

$$\vec{r}_i \equiv \vec{r}_{0i} \equiv \vec{\rho}_i - \vec{\rho}_0, \tag{2}$$

$\vec{\rho}_0$ standing for the position of the primary. The difference between

$$\ddot{\vec{\rho}}_i = \sum_{j \neq i, 0} G \frac{m_j \vec{r}_{ij}}{r_{ij}^3} + G \frac{m_0 \vec{r}_{i0}}{r_{i0}^3} \tag{3}$$

and

$$\ddot{\vec{\rho}}_0 = \sum_{j \neq i, 0} G \frac{m_j \vec{r}_{0j}}{r_{0j}^3} + G \frac{m_i \vec{r}_{0i}}{r_{0i}^3} \tag{4}$$

amounts to:

$$\ddot{\vec{r}}_i = \sum_{j \neq i, 0} G \frac{m_j \vec{r}_{ij}}{r_{ij}^3} - \sum_{j \neq i, 0} G \frac{m_j \vec{r}_j}{r_j^3} - G \frac{(m_i + m_0) \vec{r}_i}{r_i^3} = -\frac{\partial U_i}{\partial \vec{r}_i} \tag{5}$$

U_i being the potential:

$$U_i \equiv -\frac{G (m_i + m_0)}{r_i} + W_i, \tag{6}$$

with the disturbance

$$W_i \equiv -\sum_{j \neq i} G m_j \left\{ \frac{1}{r_{ij}} - \frac{\vec{r}_i \cdot \vec{r}_j}{r_j^3} \right\} \tag{7}$$

singled out. This disturbing potential acting on mass m_i is generated by the masses m_j other than m_i or the primary. It deviates from the Newtonian one by the amendment $Gm_j r_j^{-3} \vec{r}_i \cdot \vec{r}_j$ emerging in the noninertial frame associated with the primary.

In the simplest case of one secondary, a satellite of mass $m_1 = M_{\text{sat}}^*$, located at a planetocentric position $\vec{r}_1 = \vec{r}^*$, will be creating at some point $\vec{r}_2 = \vec{R}$ a perturbing potential

$$W(\vec{R}, \vec{r}^*) = -GM_{\text{sat}}^* \left\{ \frac{1}{|\vec{R} - \vec{r}^*|} - \frac{\vec{R} \cdot \vec{r}^*}{|\vec{r}^*|^3} \right\}, \tag{8}$$

expandable over the Legendre polynomials (for $R < r^*$) by means of the formulae

$$\frac{1}{|\vec{R} - \vec{r}^*|} = \frac{1}{r^*} \sum_{l=0}^{\infty} \left(\frac{R}{r^*}\right)^l P_l(\cos \gamma) \tag{9}$$

and

$$\frac{\vec{R} \cdot \vec{r}^*}{|\vec{r}^*|^3} = \frac{R r^* \cos \gamma}{r^{*3}} = \frac{1}{r^*} \frac{R}{r^*} P_1(\cos \gamma), \tag{10}$$

γ being the angular separation between $\vec{R}$ and $\vec{r}^*$, subtended at the point of origin, which we shall naturally choose to coincide with the planet's center of mass. Together, the former and the latter formulae yield:

$$W(\vec{R}, \vec{r}^*) = -\frac{GM_{\text{sat}}^*}{r^*} \sum_{l=2}^{\infty} \left(\frac{R}{r^*}\right)^l P_l(\cos \gamma), \tag{11}$$

where we have neglected the $l = 0$ term $-GM_{\text{sat}}^*/r^*$, because it bears no dependence upon $\vec{R}$, and in practical problems is attributed to the principal part of the potential, not to the one regarded as perturbation. The angle γ can be expressed via spherical coordinates as:

$$\cos \gamma = \frac{\vec{R} \cdot \vec{r}^*}{R r^*} = \sin \phi \sin \phi^* + \cos \phi \cos \phi^* \cos(\lambda - \lambda^*), \tag{12}$$

(R, ϕ, λ) being the planetocentric distance, the latitude, and the longitude of the point where the disturbance is experienced; and (r^*, ϕ^*, λ^*) being the spherical coordinates of the satellite. It is customary (though not at all obligatory) to reckon the longitudes from a planet-fixed meridian, in which case the subsequent formulae for the potential come out written in a reference frame co-rotating with the planet.

A Legendre polynomial of $\cos \gamma$, too, can be expressed via the spherical coordinates:

$$P_l(\cos \gamma) = \sum_{m=0}^{l} \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m}) P_{lm}(\sin \phi) P_{lm}(\sin \phi^*) \cos m(\lambda - \lambda^*), \tag{13}$$

substitution whereof into (11) results in

$$W(\vec{R}, \vec{r}^*) = -\frac{GM_{\text{sat}}^*}{r^*} \sum_{l=2}^{\infty} \left(\frac{R}{r^*}\right)^l \sum_{m=0}^{l} \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m})$$
$$\times P_{lm}(\sin \phi) P_{lm}(\sin \phi^*) \cos m(\lambda - \lambda^*). \tag{14}$$

Evidently, this formalism will stay unaltered, if the role of the tide-raising satellite is played by the Sun, or by another satellite, or by another planet. (In this case, what we call M_{sat} will, in

fact, denote the mass of the Sun, or of the other satellite, or of the other planet.) Likewise, the formalism may in its entirety be applied to a satellite regarded as a tidally-disturbed primary, the planet being treated as a tide-raising body (and M^*_{sat} now standing for the planetary mass).

3 The Kaula expansion for a tidal potential

Kaula (1961) came up with a remarkable formula

$$
\left(\frac{1}{r^*}\right)^{l+1} P_l(\sin\phi^*)\left[\cos m\lambda^* + \sqrt{-1}\sin m\lambda^*\right] = \left(\frac{1}{a^*}\right)^{l+1}\sum_{p=0}^{\infty} F_{\mathrm{lmp}}(i^*)
$$

$$
\sum_{q=-\infty}^{\infty} G_{\mathrm{lpq}}(e^*)\left\{\begin{array}{l}\cos\left(v^*_{\mathrm{lmpq}} - m\theta^*\right) + \sqrt{-1}\sin\left(v^*_{\mathrm{lmpq}} - m\theta^*\right)\\ \sin\left(v^*_{\mathrm{lmpq}} - m\theta^*\right) - \sqrt{-1}\cos\left(v^*_{\mathrm{lmpq}} - m\theta^*\right)\end{array}\right\}_{l-m\ \mathrm{odd}}^{l-m\ \mathrm{even}}, \quad (15)
$$

where $F_{\mathrm{lmp}}(i)$ are the inclination functions (Gooding 2008); $G_{\mathrm{lpq}}(e)$ are the eccentricity polynomials identical to the Hansen coefficients $X^{(-l-1),(l-2p)}_{(l-2p+q)}$; the notation $\sqrt{-1}$ is used to avoid confusion with the inclination; and the auxiliary combinations v^*_{lmpq} are defined as:[1]

$$
v^*_{\mathrm{lmpq}} \equiv (l-2p)\omega^* + (l-2p+q)\mathcal{M}^* + m\Omega^*. \quad (16)
$$

This development enabled Kaula (1961, 1964) to carry out a transformation from the tide-raising satellite's spherical coordinates to its orbital elements and the sidereal time θ^*. These elements (the semimajor axis a^*, eccentricity e^*, inclination i^*, periapse ω^*, ascending node Ω^*, mean anomaly $\mathcal{M}^*$) are introduced in a frame that is associated with the equator but is not co-rotating with it. In terms of these parameters,

$$
W(\vec{R}, \vec{r}^*) = -\frac{GM^*_{\mathrm{sat}}}{a^*}\sum_{l=2}^{\infty}\left(\frac{R}{a^*}\right)^l\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}(2-\delta_{0m})P_{lm}(\sin\phi)\sum_{p=0}^{l}F_{\mathrm{lmp}}(i^*)
$$

$$
\sum_{q=-\infty}^{\infty} G_{\mathrm{lpq}}(e^*)\left[\cos m\lambda\left\{\begin{array}{l}\cos\\ \sin\end{array}\right\}_{l-m\ \mathrm{odd}}^{l-m\ \mathrm{even}}\left(v^*_{\mathrm{lmpq}} - m\theta^*\right)\right.
$$

$$
\left. + \sin m\lambda\left\{\begin{array}{l}\sin\\ -\cos\end{array}\right\}_{l-m\ \mathrm{odd}}^{l-m\ \mathrm{even}}\left(v^*_{\mathrm{lmpq}} - m\theta^*\right)\right] \quad (17)
$$

or, after carrying out the multiplication of the sine and cosine functions:

$$
W(\vec{R}, \vec{r}^*) = -\frac{GM^*_{\mathrm{sat}}}{a^*}\sum_{l=2}^{\infty}\left(\frac{R}{a^*}\right)^l\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}(2-\delta_{0m})\,P_{lm}(\sin\phi)
$$

$$
\sum_{p=0}^{l} F_{\mathrm{lmp}}(i^*)\sum_{q=-\infty}^{\infty} G_{\mathrm{lpq}}(e^*)\left\{\begin{array}{l}\cos\\ \sin\end{array}\right\}_{l-m\ \mathrm{odd}}^{l-m\ \mathrm{even}}\left(v^*_{\mathrm{lmpq}} - m(\lambda+\theta^*)\right) \quad (18)
$$

[1] This definition agrees with that by Kaula (1961, 1964, 1966), but differs from the one by Lambeck (1980) who incorporated $-m\theta^*$ into v^*_{lmpq}.

4 Physical assumptions involved in Kaula's theory

If the primary is not a point mass, it becomes distorted by potential $W(\vec{R}, \vec{r}^{*})$. The distortion of shape will, in its turn, generate some extra potential perturbation whose calculation is complicated by the tide-raising potential (18) evolving in time and having a rich spectrum of frequencies. The response of the primary's shape to each of these is different and depends on the properties of the planet's material. This is a situation where the linear approach becomes most helpful, when applicable.[2]

The linear theory of bodily tides comprises two independent assertions. One is that the energy attenuation rate $\langle \dot{E} \rangle$ at each harmonic depends solely on the frequency χ and on the amplitude $E_{\text{peak}}(\chi)$, and is not influenced by the rest of the spectrum. This is written down as $\langle \dot{E}(\chi) \rangle = -\chi E_{\text{peak}}(\chi)/Q(\chi)$, which is equivalent to $\Delta E_{\text{cycle}}(\chi) = -2\pi E_{\text{peak}}(\chi)/Q(\chi)$, where $\Delta E_{\text{cycle}}(\chi)$ is the one-cycle energy loss, and $Q(\chi)$ is the quality factor. The other assertion is that each *stationary* tidal change of the potential, W_l, inflicts on the planet's shape a linear deformation. Each of these deformations, in their turn, amend the potential of the primary with an addition proportional to the Love number k_l. As known from the potential theory, an addition proportional to $P_l(\cos \gamma)$ must be decreasing outside the spherical primary as $r^{-(l+1)}$. Hence, were the external potential perturbation W static (or, equivalently, were the response of the material instant), the tidal addition to the planetary potential would have assumed the form[3]

$$
\begin{aligned}
U(\vec{r}) &= \sum_{l=2}^{\infty} k_l \left(\frac{R}{r} \right)^{l+1} W_l(\vec{R}, \vec{r}^{*}) \\
&= -\sum_{l=2}^{\infty} k_l \left(\frac{R}{r} \right)^{l+1} \frac{GM_{\text{sat}}^{*}}{a^{*}} \left(\frac{R}{a^{*}} \right)^{l} \sum_{m=0}^{l} \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m}) P_{lm}(\sin \phi) \\
&\quad \sum_{p=0}^{l} F_{lmp}(i^{*}) \sum_{q=-\infty}^{\infty} G_{lpq}(e^{*}) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix}_{\substack{l-m \text{ even} \\ l-m \text{ odd}}} \left(v_{lmpq}^{*} - m(\lambda + \theta^{*}) \right)
\end{aligned} \tag{19}
$$

R being the mean equatorial (equivolumetric) radius of the planet, $\vec{R} = (R, \phi, \lambda)$ being a particular surface point, and $\vec{r} = (r, \phi, \lambda)$ being an exterior point located right above the surface point $\vec{R}$, at a planetocentric radius $r \geq R$.

As we intend to study the effect of this potential on another external body, a similar transformation should be applied to the coordinates (r, ϕ, λ), to express W through the orbital elements of this body. Employment of (15), this time not for $\vec{r}^{*}$ but for $\vec{r}$, leads to:

---

[2]   For most materials, departure from linearity becomes considerable when the strains approach 10^{-6}. (Karato 2007)

[3]   Following MacDonald (1964) and Singer (1968), we denote the tide-raising potential with W and the bodily-tide one with U. In his original paper, Kaula (1964) called these potentials U and T, while in the book he switched to U and U_T (Kaula 1968). Be mindful that we are using a sign convention different from that of Kaula. As our forces are negative gradients of potentials, our potentials are negative to those of Kaula.

$$U(\vec{r}) = -\sum_{l=2}^{\infty} k_l \left(\frac{R}{a}\right)^{l+1} \frac{GM_{\text{sat}}^*}{a^*} \left(\frac{R}{a^*}\right)^l \sum_{m=0}^{l} \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m})$$

$$\sum_{p=0}^{l} F_{\text{lmp}}(i^*) \sum_{q=-\infty}^{\infty} G_{\text{lpq}}(e^*) \sum_{h=0}^{l} F_{\text{lmh}}(i)$$

$$\sum_{j=-\infty}^{\infty} G_{\text{lhj}}(e) \cos\left[\left(v_{\text{lmpq}}^* - m\theta^*\right) - \left(v_{\text{lmhj}} - m\theta\right)\right], \tag{20}$$

a formula that generalises the tidal theory of Darwin (1908, p. 334) to l and $|q|$ larger than 2. Both Kaula (1964), who derived this milestone result, and Darwin, who had developed its simplified version, realised that this machinery would work only after the material's delayed reaction to perturbation (18) is somehow taken into account. Until then (20) remains idealised, in that it corresponds to an unphysical case of instantaneous response.

To account for damping, Kaula (1964) followed the path of Darwin (1880, 1908): he endowed each term of the Fourier series (20) with a real phase lag of its own, ϵ_{lmpq}, where-after the ultimate form of Kaula's expansion became

$$U(\vec{r}) = -\sum_{l=2}^{\infty} k_l \left(\frac{R}{a}\right)^{l+1} \frac{GM_{\text{sat}}^*}{a^*} \left(\frac{R}{a^*}\right)^l \sum_{m=0}^{l} \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m})$$

$$\sum_{p=0}^{l} F_{\text{lmp}}(i^*) \sum_{q=-\infty}^{\infty} G_{\text{lpq}}(e^*) \sum_{h=0}^{l} F_{\text{lmh}}(i)$$

$$\sum_{j=-\infty}^{\infty} G_{\text{lhj}}(e) \cos\left[\left(v_{\text{lmpq}}^* - m\theta^*\right) - \left(v_{\text{lmhj}} - m\theta\right) - \epsilon_{\text{lmpq}}\right]. \tag{21}$$

This empirical method of including dissipation into the picture contains in itself an important omission, of which Sir George Darwin was aware, but which was overlooked by his successors. Briefly speaking, even in a linear system a dissipation process is *not* fully accounted for by amending phases of the Fourier components. This observation happens to be of relevance in the theory of tidal torques. We shall return to this point in Sect. 9.

5 The two sidereal angles

Kaula's construction contains a seemingly redundant fixture, which turns out to be an important and useful acquisition. This is Kaula's introducing two sidereal angles instead of one. As these angles, θ and θ^*, are not orbital elements of the tide-raising and tidally disturbed moons, but are parameters characterizing the instantaneous attitude of the planet, it may look strange that Kaula (1964) assumed them to be different entities. To understand his point, let us trace the physical origin of the phase lag. The material of the primary is being deformed by a tidal stress whose spectrum contains an infinite number of frequencies, the reaction of the material to each of these being different. In a linear regime, the strain has the same spectrum, with each harmonic delayed by its own time lag Δt_{lmpq}. Singer (1968), and later Mignard (1979, 1980), assumed that all Δt_{lmpq} are equal to one another: $\Delta t_{\text{lmpq}} = \Delta t$. If this were true, then in Kaula's series each argument $v_{\text{lmpq}}^* - m\theta^*$ would have to be substituted with

$$
\begin{aligned}
v^{*(\text{delayed})}_{\text{lmpq}} - m\theta^{*(\text{delayed})} &\equiv v^*_{\text{lmpq}}(t - \Delta t) - m\theta^*(t - \Delta t) \\
&= v^*_{\text{lmpq}}(t) - m\theta^*(t) - \left[\dot{v}^*_{\text{lmpq}} - m\dot{\theta}^* \right] \Delta t \\
&= v^*_{\text{lmpq}}(t) - m\theta^*(t) - \left[(l - 2p)\dot{\omega}^* + (l - 2p + q)\dot{\mathcal{M}}^* \right. \\
&\quad \left. + m(\dot{\Omega}^* - \dot{\theta}^*) \right] \Delta t.
\end{aligned}
\tag{22}
$$

In reality, however, the time lag is a function of frequency, for which reason the delays Δt_{lmpq} will be different for each harmonic involved. This is why the arguments $v^*_{\text{lmpq}} - m\theta^*$ at the moment t should rather be replaced with

$$
\begin{aligned}
v^{*(\text{delayed})}_{\text{lmpq}} - m\theta^{*(\text{delayed})}_{\text{lmpq}} &= v^*_{\text{lmpq}} - m\theta^* - \left[(l - 2p)\dot{\omega}^* \right. \\
&\quad \left. + (l - 2p + q)\dot{\mathcal{M}}^* + m(\dot{\Omega}^* - \dot{\theta}^*) \right] \Delta t_{\text{lmpq}}.
\end{aligned}
\tag{23}
$$

Specifically,

$$
v^{*(\text{delayed})}_{\text{lmpq}} = v^*_{\text{lmpq}} - \left[(l - 2p)\dot{\omega}^* + (l - 2p + q)\dot{\mathcal{M}}^* + m\dot{\Omega}^* \right] \Delta t_{\text{lmpq}}
$$

and

$$
\theta^{*(\text{delayed})}_{\text{lmpq}} = \theta^* - \dot{\theta}^* \Delta t_{\text{lmpq}},
$$

$\dot{\theta}^*$ being the planet spin rate. In brief, (23) can be rewritten as

$$
v^{*(\text{delayed})}_{\text{lmpq}} - m\theta^{*(\text{delayed})}_{\text{lmpq}} = v^*_{\text{lmpq}} - m\theta^* - \omega_{\text{lmpq}} \Delta t_{\text{lmpq}}.
$$

We see that the total phase lags ϵ_{lmpq} introduced by Kaula are given by

$$
\begin{aligned}
\epsilon_{\text{lmpq}} &= \left[(l - 2p)\dot{\omega}^* + (l - 2p + q)\dot{\mathcal{M}}^* + m(\dot{\Omega}^* - \dot{\theta}^*) \right] \Delta t_{\text{lmpq}} \\
&= \omega^*_{\text{lmpq}} \Delta t_{\text{lmpq}} = \pm \chi^*_{\text{lmpq}} \Delta t_{\text{lmpq}},
\end{aligned}
\tag{24}
$$

the tidal harmonic ω^*_{lmpq} being introduced as

$$
\omega^*_{\text{lmpq}} \equiv (l - 2p)\dot{\omega}^* + (l - 2p + q)\dot{\mathcal{M}}^* + m(\dot{\Omega}^* - \dot{\theta}^*),
\tag{25}
$$

the positively-defined physical frequency

$$
\chi^*_{\text{lmpq}} \equiv |\omega^*_{\text{lmpq}}| = |(l - 2p)\dot{\omega}^* + (l - 2p + q)\dot{\mathcal{M}}^* + m(\dot{\Omega}^* - \dot{\theta}^*)|
\tag{26}
$$

being the actual physical *lmpq* tidal frequency excited in the primary's material. The appropriate positively-defined time delay Δt_{lmpq} depends on this physical frequency, for which reason the delays Δt_{lmpq} are, generally, different from one another.[4]

The sign on the right-hand side of (24) is simply the sign of ω^*_{lmpq}. The sign evidently depends on whether $m\dot{\theta}$ falls short of or exceeds the linear combination $(l - 2p)\dot{\omega}^* + (l - 2p + q)\dot{\mathcal{M}}^* + m\dot{\Omega}^* \approx (l - 2p + q)\dot{\mathcal{M}}^*$.

The origin and meaning of the phase lag ϵ_{lmpq} being now transparent, one may express the cosine functions in (21) either as

$$
\cos\left[\left(v^*_{\text{lmpq}} - m\theta^* \right) - (v_{\text{lmhj}} - m\theta) - \epsilon_{\text{lmpq}} \right]
\tag{27}
$$

---

[4] When Kaula was developing his theory, the functional form of the dependence $\Delta t(\chi)$ was not yet known. Reliable data became available only in the final quarter of the past century. See formula (83) below.

(where θ^* and θ are identical and cancel one another), or simply as

$$\cos\left[\left(v_{\text{lmpq}}^{*(\text{delayed})} - m\theta_{\text{lmpq}}^{*(\text{delayed})}\right) - \left(v_{\text{lmhj}} - m\theta\right)\right]. \tag{28}$$

In (28) we have the delayed siderial angle, $\theta_{\text{lmpq}}^{*(\text{delayed})}$, separated from the actual angle, θ, by $-\dot{\theta}\Delta t_{\text{lmpq}}$, the time lag Δt_{lmpq} being a function of $\chi_{\text{lmpq}} \equiv |\omega_{\text{lmpq}}|$.

6 The Darwin–Kaula–Goldreich expansion for the tidal torque

Now we are prepared to calculate the planet-perturbing tidal torque. Since in what follows we shall dwell on the low-inclination case, it will be sufficient to derive the torque's component orthogonal to the planetary equator:

$$\tau = -M_{\text{sat}}\frac{\partial U(\vec{r})}{\partial \theta}, \tag{29}$$

M_{sat} being the mass of the tide-disturbed satellite, and the "minus" sign emerging due to our choice not of the astronomical but of the physical sign convention. Adoption of the latter convention implies the emergence of a "minus" sign in the expression for the potential of a point mass: $-GM/r$. This "minus" sign then shows up on the right-hand sides of (6–8) and, later, of (19–21). It is then compensated by the "minus" sign standing in (29).

The right way of calculating $\partial U(\vec{r})/\partial\theta$ is to take the derivative of (28) with respect to θ, then to insert (23) into the result, and finally to get rid of the sidereal angle completely, by imposing the constraint $\theta^* = \theta$. This will yield:[5]

$$\tau = -\sum_{l=2}^{\infty} k_l \left(\frac{R}{a}\right)^{l+1} \frac{GM_{\text{sat}}^* M_{\text{sat}}}{a^*}\left(\frac{R}{a^*}\right)^l \sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}2m$$

$$\sum_{p=0}^{l} F_{\text{lmp}}(i^*)\sum_{q=-\infty}^{\infty} G_{\text{lpq}}(e^*)\sum_{h=0}^{l} F_{\text{lmh}}(i)$$

$$\sum_{j=-\infty}^{\infty} G_{\text{lhj}}(e)\sin\left[v_{\text{lmpq}}^* - v_{\text{lmhj}} - \epsilon_{\text{lmpq}}\right], \tag{30}$$

In the case of the tide-raising satellite coinciding with the tide-perturbed one, $M_{\text{sat}} = M_{\text{sat}}^*$, and all the elements become identical to their counterparts with an asterisk. For a primary body not in a tidal lock with its satellite,[6] the torque (30) can be split into two parts. The

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[5] Formally, one can as well differentiate (27) instead of (28), first ignoring the fact that θ^* and θ are identical and then, after differentiation, permitting them to cancel one another. Though this method produces the same result as the rigorous calculation, it nonetheless remains a formal procedure lacking physics in it.

[6] This caveat is relevant, because in resonances expressions (33–35) will require modifications. For example, the sidereal angle of a satellite tidally locked in a $1:1$ resonance will be: $\theta = \Omega + \omega + \mathcal{M} + 180° + \alpha + O(i^2)$, letter α denoting the librating angle, which is subject to damping and therefore is normally small (less than 2" for the Moon). Inserting the said formula for θ into the expression (25) for the tidal harmonic, we obtain, in neglect of $-m\dot{\alpha}$:

$$\omega_{\text{lmpq}}^* \equiv (l-2p-m)\dot{\omega}^* + (l-2p+q-m)\dot{\mathcal{M}}^*.$$

We now see that, since θ is a function of the other angles, different sets of the indices's values will correspond to one value of the tidal frequency. We shall illustrate this by considering the so-called anomalistic modes $\pm\mathcal{M}$ in the potential. These modes, corresponding to the physical frequency $|\mathcal{M}|$, are given by $(lmpq) = (201, \pm1)$

first part is constituted by those terms of (30), in which indices $(p,\ q)$ coincide with $(h,\ j)$, and therefore all v_{lmhj} cancel with v^*_{lmpq}, provided the tidally-perturbed satellite and the tide-raising one are the same body. This component of the torque is, therefore, constant. The rest of the total sum (30) will be denoted with $\tilde{\tau}$. It is comprised of the terms, in which the pairs $(p,\ q)$ differ from $(h,\ j)$. Accordingly, these terms contain the differences

$$v^*_{\text{lmpq}} - v_{\text{lmhj}} = (l - 2p + q)\mathcal{M}^* - (l - 2h + j)\mathcal{M} + m(\Omega^* - \Omega)$$
$$+ l(\omega^* - \omega) - 2p\omega^* + 2h\omega. \tag{31}$$

When the tidally-perturbed and tide-raising moons are the same body, this becomes

$$v^*_{\text{lmpq}} - v_{\text{lmhj}} = (2h - 2p + q - j)\mathcal{M} + (2h - 2p)\omega, \tag{32}$$

whence we see that the oscillating component of the torque, $\tilde{\tau}$, consists of two parts. The part with $h - p = 0$ and $q - j \neq 0$ consists solely of short-period terms, and it averages out trivially.

The mixed-period part of (30), with $h - p \neq 0$, consists of both a short-period contribution dependent upon the mean anomaly, and a long-period contribution depending upon the argument of pericentre. All such terms contain multipliers like $F_{2mp}(i) F_{2mh}(i)$, where $h \neq p$, and $F_{220} = 3 + O(i^2)$, $F_{210} = 3/2 \sin i + O(i^3)$, $F_{211} = -3/2 \sin i + O(i^3)$, $F_{221} = 3/2 \sin^2 i$, the other relevant F_{2mn}'s being of higher order than $O(i^2)$. So the only long-period terms that we have to consider in (30) involve products: $F_{210}(i)F_{211}(i)$, $F_{211}(i)F_{210}(i)$, $F_{220}(i)F_{221}(i)$, and $F_{221}(i)F_{220}(i)$. However, these products are of order $O(i^2)$. Thus, while the short-period terms in (30) average out over one rotation period of the moon about the planet, the long-period terms are of order $O(e^2 i^2)$, the e^2 coming from the G_{2pq} functions. (Indeed, when h and p differ by 1, then q and j must differ by 2, to eliminate the mean anomaly, i.e., to make the term long-period and not short-period.) So both the short- and long-period contributions may be neglected in our approximation.[7]

Footnote 6 continued

and also by $(lmpq) = (220, \pm 1)$. Although the $m = 0$ terms enter the potential, they will not be in the torque, as can be observed by differentiating (19) with respect to λ, or by differentiating equation (21) with respect to $-\theta$, or simply by noticing the presence of the factor m on the right-hand side of (30). Nonetheless, we see that there exists a pair of $m = 2$ terms, which provides an anomalistic input into the torque. This way, the case of libration deserves a separate consideration, as it is more involved than that of tidal despinning. Specifically, in the case of libration a value of the tidal frequency may correspond to different sets of the indices' values.

[7] Had we tried to expand our treatment to higher inclinations, our neglect of the short-period terms would remain legitimate, for they still would average out over one rotation period of the satellite about its primary. As for the long-period terms, it would be tempting to say that these average out over the apsidal-precession period. The latter is much shorter than the time scale of the planetary spin deceleration, a circumstance that may seem a safe justification for the neglect of the long-period terms also for higher inclinations. However, a word of warning would be appropriate here. As well known from Kozai (1959a), who took into account the primary's nonsphericity, the pericentre of a satellite inclined by about $63°$ or $117°$ will neither advance nor retard, at least within the first-order (in J_2) perturbation theory. (For a critical review of Kozai's theory see Taff (1985).) Kozai's original attempt to introduce corrections owing to J_3 and J_4 was flawed because in the vicinity of the critical inclinations these terms should be considered not as higher-order but rather as leading. His later analysis demonstrated that at these inclinations the satellite's perigee should librate about $90°$ or $270°$ (Kozai 1962). Under these circumstances, the long-period terms in our expression for the torque will *not* be averaged out. We, however, may neglect this possibility, because in the current work we consider only low-inclined moons.

Another situation, which we exclude from our treatment, is libration of the satellite's periapse about $90°$ or $270°$, caused by the pull of a third body (the star or some large neighboring planet). The possibility of such librations may be derived from the presence of the $\cos 2\omega$ term on the right-hand side of the equation for $d\omega/dt$ in the theory of Kozai (1959b, 1962)—for an easy introduction into this theory see Innanen et al. (1997), and for its generalisation to finite obliquities see Gurfil et al. (2007). An important special case of

Thus, we arrive at:

$$\tau = \sum_{l=2}^{\infty} 2k_l GM_{\text{sat}}^2 \frac{R^{2l+1}}{a^{2l+2}} \sum_{m=0}^{l} \frac{(l-m)!}{(l+m)!} m \sum_{p=0}^{l} F_{\text{lmp}}^2(i) \sum_{q=-\infty}^{\infty} G_{\text{lpq}}^2(e) \sin \epsilon_{\text{lmpq}} + \tilde{\tau}, \quad (33)$$

the sum standing for the constant ($\mathcal{M}$-independent) part of the torque, and $\tilde{\tau}$ denoting the oscillating part whose time-average is zero.

As we pointed in the end of Sect. 5, the sign of the phase lag ϵ_{lmpq} depends on whether $m\dot{\theta}$ falls short of or exceeds the linear combination $(l-2p)\dot{\omega}^* + (l-2p+q)\dot{\mathcal{M}}^* + m\dot{\Omega}^* \approx (l-2p+q)\dot{\mathcal{M}}^*$. Now we also understand that, outside resonances, the $lmpq$ component of the tidal torque experienced by the planet is decelerating if the values of $m\dot{\theta}$ exceed the given combination, and is accelerating otherwise.

Expression (33) gets considerably simplified if we restrict ourselves to the case of $l = 2$. Since $0 \leq m \leq l$, and since m enters the expansion as a multiplier, we see that only $m = 1, 2$ actually matter. As $0 \leq p \leq l$, we are left with only six relevant F's, those corresponding to $(lmp) = (210), (211), (212), (220), (221)$, and (222). By a direct inspection of the table of F_{lmp} we find that five of these six functions happen to be $O(i)$ or $O(i^2)$, the sixth one being $F_{220} = \frac{3}{4}(1 + \cos i)^2 = 3 + O(i^2)$. Thus, in the leading order of i, the constant part of the torque reads:

$$\tau_{l=2} = \frac{3}{2} \sum_{q=-\infty}^{\infty} GM_{\text{sat}}^2 R^5 a^{-6} G_{20q}^2(e) k_2 \sin \epsilon_{220q} + O(i^2/Q). \quad (34)$$

This is what is called Darwin–Kaula–Goldreich torque, or simply Darwin torque. The principal term of this series is

$$\tau_{2200} = \frac{3}{2} GM_{\text{sat}}^2 k_2 R^5 a^{-6} \sin \epsilon_{2200}. \quad (35)$$

Switching from the lags to quality factors via formula[8]

$$Q_{\text{lmpq}} = |\cot \epsilon_{\text{lmpq}}|, \quad (36)$$

Footnote 7 continued

the theory of the third-body-caused librations is the one of the satellite getting into a resonance with the third body. An indication that such resonances may cause the satellite's ω librate comes from the mathematically similar theory of Pluto–Neptune resonances (Williams and Benson 1971): being in resonance with Neptune, Pluto has its periapse librating due to a high inclination. (To be exact, the behaviour of Pluto's periapse is dictated not only by Neptune, but by the combined influence of all of the four gas giant planets. However, this does not change the main point: the outer body or bodies can cause apsidal libration.)

In the Solar system, none of the large satellites is so highly perturbed as to have a periapse librating around $90°$ or $270°$ due to the above two mechanisms. In theory, though, this remains an option for exoplanets. Either librating mechanism might apply also to satellites of minor planets. In our current paper we do not consider such moons.

[8] The phase lag ϵ_{lmpq} is introduced in (23–24), while the tidal harmonic ω_{lmpq} is given by (25). The quality factor $Q_{\text{lmpq}} = |\cot \epsilon_{\text{lmpq}}|$ is, for physical reasons, positively defined. Hence the multiplier sgn ω_{lmpq} in (37). (As ever, the function sgn (x) is defined to assume the values $+1$, -1, or 0 for positive, negative, or vanishing x, correspondingly.)

Mind that no factor of two appears in (36–37), because ϵ is a phase lag, not a geometric angle.

we obtain:

$$\sin \epsilon_{lmpq} = \sin |\epsilon_{lmpq}| \text{sgn } \omega_{lmpq} = \frac{\text{sgn } \omega_{lmpq}}{\sqrt{1 + \cot^2 \epsilon_{lmpq}}} = \frac{\text{sgn } \omega_{lmpq}}{\sqrt{1 + Q_{lmpq}^2}}$$

$$= \frac{\text{sgn } \omega_{lmpq}}{Q_{lmpq}} + O(Q^{-3}), \qquad (37)$$

whence

$$\tau_{l=2} = \frac{3}{2} \sum_{q=-\infty}^{\infty} GM_{\text{sat}}^2 R^5 a^{-6} G_{20q}^2(e) k_2 \frac{\text{sgn } \omega_{220q}}{Q_{220q}} + O(i^2/Q) + O(Q^{-3}).$$

Now, let us simplify the sign multiplier. If in expression (25) for ω_{lmpq} we get rid of the redundant asterisks, replace[9] $\dot{\mathcal{M}}$ with $\dot{\mathcal{M}}_0 + n \approx n$, and set $l = m = 2$ and $p = 0$, the outcome will be:

$$\text{sgn } \omega_{220q} = \text{sgn } \left[2\dot{\omega} + (2 + q)n + 2\dot{\Omega} - 2\dot{\theta} \right]$$

$$= \text{sgn } \left[\dot{\omega} + \left(1 + \frac{q}{2} \right) n + \dot{\Omega} - \dot{\theta} \right].$$

As the node and periapse precessions are slow, the above expression may be simplified to

$$\text{sgn } \left[\left(1 + \frac{q}{2} \right) n - \dot{\theta} \right].$$

All in all, the approximation for the constant part of the torque assumes the form:

$$\tau_{l=2} = \frac{3}{2} \sum_{q=-\infty}^{\infty} GM_{\text{sat}}^2 R^5 a^{-6} G_{20q}^2(e) k_2 Q_{220q}^{-1}$$

$$\times \text{sgn } \left[\left(1 + \frac{q}{2} \right) n - \dot{\theta} \right] + O(i^2/Q) + O(Q^{-3}). \qquad (38)$$

That the sign of the right-hand side in the above formula is correct can be checked through the following obvious observation: for a sufficiently high spin rate $\dot{\theta}$ of the planet, the multiplier $\text{sgn } \left[\left(1 + \frac{q}{2} \right) n - \dot{\theta} \right]$ becomes negative. Thereby the overall expression for $\tau_{l=2}$ acquires a "minus" sign, so that the torque points out in the direction of rotation opposite to the direction of increase of the sidereal angle θ. This is exactly how it should be, because for a fixed q and a sufficiently fast spin the q's component of the tidal torque must be decelerating and driving the planet to synchronous rotation.

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[9] While in the undisturbed two-body setting $\mathcal{M} = \mathcal{M}_0 + n(t - t_0)$ and $\dot{\mathcal{M}} = n$, under perturbation these relations get altered. One possibility is to introduce (following Tisserand 1896) an *osculating mean motion* $n(t) \equiv \sqrt{\mu/a(t)^3}$, and to stick to this definition under perturbation. Then the mean anomaly will evolve as $\mathcal{M} = \mathcal{M}_0 + \int_{t_0} n(t)dt$, whence $\dot{\mathcal{M}} = \dot{\mathcal{M}}_0(t) + n(t)$.

Other possibilities include introducing an *apparent* mean motion, i.e., defining n either as the mean-anomaly rate $d\mathcal{M}/dt$, or as the mean-longitude rate $dL/dt = d\Omega/dt + d\omega/dt + d\mathcal{M}/dt$ (as was done by Williams et al. 2001). It should be mentioned that, while the first-order perturbations in $a(t)$ and in the osculating mean motion $\sqrt{\mu/a(t)^3}$ do not have constant parts leading to secular rates, the epoch terms typically do have secular rates. These considerations explain why there exists a difference between the apparent mean motion defined as dL/dt (or as $d\mathcal{M}/dt$) and the osculating mean motion $\sqrt{\mu/a(t)^3}$.

In many practical situations, the secular rate in $\mathcal{M}_0$ is of the order of the periapse rate, while the secular rate in L_0 turns out to be smaller. Hence the advantage of defining the apparent n as the mean-longitude rate dL/dt, rather than as the mean-anomaly rate $d\mathcal{M}/dt$. (At the same time, for a satellite orbiting an oblate planet the secular rates of $\mathcal{M}_0$, L_0, and periapse are of the same order.)

Although the causes of orbit perturbations are beyond the scope of our paper, we would mention that in the expression (25) for ω_{lmpq} the notations $\dot{\mathcal{M}}, \dot{\omega}$, and $\dot{\Omega}$ generally imply the *secular* rate.

Expansion (38) was written down for the first time, without proof, by Goldreich and Peale (1996). A schematic proof was later offered by Dobrovolskis (2007).

7 The MacDonald expression for the tidal torque

The idea of representing the tidal pattern with one bulge is often mis-attributed to MacDonald (1964) who in fact borrowed it from Gerstenkorn (1955). This approach was furthered by Singer (1968) and greatly advanced by Mignard (1979, 1980). It is the latter two authors who realised that Gerstenkorn's single-bulge simplification was acceptable only with a frequency-independent Δt, not with a frequency-independent Q as in MacDonald (1964). Nevertheless we shall call this approach "the MacDonald torque", to comply with the established convention. For the same reason, the afore-described Darwin–Kaula–Goldreich expansion will be referred to simply as "the Darwin torque".

In the preceding section, the Darwin torque's component orthogonal to the equator was conveniently given by the fundamental formula (29). Within the MacDonald approach, it will be more practical to write the torque as a derivative taken with respect to the longitude. The torque acting on the tidally disturbed satellite of mass M_{sat} is $-M_{sat}\partial U/\partial\lambda$, while the torque that this moon exerts on the planet is this expression's negative:

$$\tau(\vec{r}) = M_{sat}\frac{\partial U(\vec{r})}{\partial\lambda}. \tag{39}$$

Speaking rigorously, the formula furnishes the torque's component perpendicular to the planetary equator. As can be seen from (44), formula (39) coincides with (29) for low inclinations.

7.1 Simplifications available for low i

In principle, we can as well insert into

$$U(\vec{r}) = \sum_{l=2}^{\infty} k_l \left(\frac{R}{r}\right)^{l+1} W_l(\vec{R},\vec{r}^*)$$

the "raw" expression (14), the one as yet "unprocessed" by (15). This will give us

$$U(\vec{r}) = -GM_{sat}^* \sum_{l=2}^{\infty} k_l \frac{R^{2l+1}}{r^{l+1}r^{*l+1}} \sum_{m=0}^{l} \frac{(l-m)!}{(l+m)!}(2-\delta_{0m})$$
$$\times P_{lm}(\sin\phi)P_{lm}(\sin\phi^*)\cos m(\lambda-\lambda^*) \tag{40}$$

or, for low inclinations of both the tidally-perturbed and tide-raising satellites:

$$U(\vec{r}) = -GM_{sat}^* \sum_{l=2}^{\infty} k_l \frac{R^{2l+1}}{r^{l+1}r^{*l+1}} \sum_{m=0}^{l} \frac{(l-m)!}{(l+m)!}(2-\delta_{0m})P_{lm}(0)$$
$$\times P_{lm}(0)\cos m(\lambda-\lambda^*) + O(i^2) + O(i^{*2}) + O(ii^*). \tag{41}$$

At this point, we once again are faced with the question of how to bring damping into the picture, i.e., how to take care of the delayed reaction of the planet's material to the tidal stress. It is tempting to substitute $m\lambda^*$ with its delayed value. Then instead of $\cos m(\lambda-\lambda^*)$ we get

$$\cos\left(m\lambda - m\lambda^{*(delayed)}\right) = \cos\left(m\lambda - \left[m\lambda^* - m\dot{\lambda}^*\Delta t\right]\right), \tag{42}$$

This trick, suggested by Kaula (1968, p. 201),[10] has a physical justification only if Δt is the same for all frequencies, a model pioneered by Singer (1968) and furthered by Mignard (1979, 1980). It can be shown that this model is equivalent to the following rheological law:[11]

$$Q_{\text{lmpq}} = \frac{1}{\chi_{\text{lmpq}}\,\Delta t}. \tag{43}$$

Even then, though, it remains unclear how to connect the longitude lag $m\,\dot{\lambda}^*\Delta t$ with one or another Q_{lmpq}, in the spirit of (37). To see what can be done, write down the longitude (reckoned from a fixed meridian on the rotating planet) as

$$\lambda = -\theta + \Omega + \omega + \nu + O(i^2) = -\theta + \Omega + \omega + \mathcal{M}$$
$$+2e\sin\mathcal{M} + O(e^2) + O(i^2), \tag{44}$$

ν being the true anomaly. Thence, in neglect of the nodal and apsidal precessions, the cosine becomes:

$$\cos\left(\left[m\lambda - m\lambda^*\right] + m\,\dot{\lambda}^*\Delta t\right) = \cos\left(\left[m\lambda - m\lambda^*\right] + m\left[\dot{\nu}^* - \dot{\theta}^*\right]\Delta t\right), \tag{45}$$

or, equivalently:

$$\cos\left(\left[m\lambda - m\lambda^*\right] + m\,\dot{\lambda}^*\Delta t\right) = \cos\left(\left[m\lambda - m\lambda^*\right] + m\left[n^* - \dot{\theta}^*\right]\Delta t\right.$$
$$\left. + 2me^*n^*\Delta t\cos\mathcal{M}^* + O(e^2)\right). \tag{46}$$

Insertion of (45) into (41), along with substitution of $r^*(t)$ by $r^*(t - \Delta t)$, leads us to

$$U(\vec{r}) = -GM_{\text{sat}}^* \sum_{l=2}^{\infty} k_l \frac{R^{2l+1}}{r(t)^{l+1}r^*(t-\Delta t)^{l+1}} \sum_{m=0}^{l} \frac{(l-m)!}{(l+m)!}(2-\delta_{0m})P_{lm}(0)$$
$$\times P_{lm}(0)\cos\left(m\left[\lambda - \lambda^*\right] + m\left[\dot{\nu}^* - \dot{\theta}^*\right]\Delta t\right) + O(i^2) + O(i^{*2}) + O(ii^*). \tag{47}$$

If we take into account only the $l = 2$ contribution, expression (41) will simplify to

$$U(\vec{r}) = -\frac{GM_{\text{sat}}^* k_2 R^5}{r(t)^3 r^*(t)^3} \sum_{m=0}^{2} \frac{(2-m)!}{(2+m)!}(2-\delta_{0m})P_{2m}(0)P_{2m}(0)$$
$$\times \cos m(\lambda - \lambda^*) + O(i^2) + O(i^{*2}) + O(ii^*), \tag{48}$$

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[10] Mind the difference in notations. While in the original paper Kaula (1964) denoted the phase lags with ϵ_{lmpq}, in his book Kaula (1968) called them φ_{lmpq}. For the longitudinal lag $2\,\dot{\lambda}^*\Delta t$ emerging in our formula (46), Kaula (1968) used notation 2δ. This way, in the terms used by Kaula (1968) in his book, the geometric angle subtended at the primary's center between the directions to the bulge and the moon is called 2δ, not δ as in most literature.

[11] Combining (24) with the relation $Q = 1/\tan|\epsilon|$, we see that setting all Δt_{lmpq} equal to the same Δt is equivalent to saying that the quality factor scales as the inverse frequency: $Q = 1/(\chi\Delta t)$, provided, of course, that the Q factor is large. As can be seen from (37), a more exact relation will read: $\sin(\chi\Delta t) = 1/\sqrt{1+Q^2}$, so that $\chi\Delta t = Q^{-1} + O(Q^{-3})$.
  Very special is the case when the values of the quality factor are very low (say, much less than 10). In this situation, the interconnection between the quality factor and the phase lag becomes quite different from the customary formula $Q = \cot|\epsilon|$. See the Appendix for details.

where only the $m = 2$ term is important.[12] In the presence of dissipation, the appropriately simplified version of (48) will read:

$$U(\vec{r}) = -\frac{3}{4} \frac{GM_{\text{sat}}^* k_2 R^5}{r(t)^3 r^*(t-\Delta t)^3} \cos\left(\left[2\lambda - 2\lambda^*\right] + 2\left[\dot{\nu}^* - \dot{\theta}^*\right]\Delta t\right)$$
$$+ O(i^2) + O(i^{*2}) + O(ii^*), \tag{49}$$

while the corresponding expression for the torque exerted by the satellite on the planet will, in this approximation, be given by

$$\tau(\vec{r}) = M_{\text{sat}} \frac{\partial U(\vec{r})}{\partial \lambda} = \frac{3}{2} \frac{GM_{\text{sat}}^* M_{\text{sat}} k_2 R^5}{r(t)^3 r^*(t-\Delta t)^3} \sin\left(\left[2\lambda - 2\lambda^*\right] + 2\left[\dot{\nu}^* - \dot{\theta}^*\right]\Delta t\right)$$
$$+ O(i^2) + O(i^{*2}) + O(ii^*). \tag{50}$$

In the case when the tidally disturbed satellite coincides with the tide-raising one, i.e., when $\lambda = \lambda^*$ and $M_{\text{sat}} = M_{\text{sat}}^*$, we obtain:

$$\tau = \frac{3}{2} GM_{\text{sat}}^2 k_2 \frac{R^5}{r(t)^3 r(t-\Delta t)^3} \sin\left(2\left[\dot{\nu} - \dot{\theta}\right]\Delta t\right) + O(i^2/Q)$$
$$= \frac{3}{2} GM_{\text{sat}}^2 k_2 \frac{R^5}{r^6} \sin\left(2\left[\dot{\nu} - \dot{\theta}\right]\Delta t\right) + O(i^2/Q) + O(en/Q^2\chi), \tag{51}$$

where the error $O(en/Q^2\chi)$ emerges when we identify the lagging distance $r(t - \Delta t)$ with $r \equiv r(t)$. Replacement of $r(t - \Delta t)$ with r is convenient, though not necessary. In Subsect. 7.2 below, we shall explain that, after averaging over one revolution of the moon about the planet, the error caused by this replacement reduces to $O(e^2 n^2/Q^3\chi^2)$, which will be less than the largest error.

The MacDonald torque (51) is equivalent to the Darwin torque (34) with an important proviso that all time lags Δt_{lmpq} are equal to one another or, equivalently, that the rheological model (43) is accepted. Physically, the special case of equal time lags is exactly the case when the tide may be rigorously interpreted as one double bulge of a variable rate and amplitude.[13] Mathematically, this model enables one to wrap up the infinite series (34) into the elegant finite form (51). Formally, this wrapping can be described like this: expression (51) mimics the principal term of the series (34), provided in this term the multiplier G_{200}^2 is replaced with unity, a is replaced with r, and the principal phase lag

$$\epsilon_{2200} \equiv 2(n - \dot{\theta})\Delta t \tag{52}$$

is replaced with the longitudinal lag or, possibly better to say, with the quasi-phase

$$\epsilon \equiv 2(\dot{\nu} - \dot{\theta})\Delta t. \tag{53}$$

Thus, we see that within the MacDonald one-variable-bulge formalism the longitudinal lag (53) is acting as an *instantaneous* phase lag associated with the *instantaneous* tidal frequency $\chi \equiv 2|\dot{\nu} - \dot{\theta}|$. This is why we may call it simply ϵ, without a subscript. Evidently, ϵ is (up to

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[12] In (48), we may neglect the λ-independent term with $m = 0$, because our eventual intention is to find the torque by differentiating $U(\vec{r})$ with respect to λ. We may also omit the $m = 1$ term, because $P_{21}(0) = 0$. This omission brings up an error of order $O(ii^*)$ into equations (41), (47–50) and (56).

[13] An attempt to generalise this simplified approach to arbitrary inclinations was undertaken by Efroimsky (2006). While for constant time lags that generalisation is likely to be acceptable, it remains to be explored whether the offers a practical approximation for actual rheologies (60).

a sign) twice the geometrical angle subtended at the primary's center between the directions to the moon and to the bulge.[14]

The geometric meaning of the longitudinal lag being clear, let us consider its physical meaning, in the sense of this lag's relation to the dissipation rate. For some fixed frequency χ_{lmpq}, the corresponding phase lag ϵ_{lmpq} is related to the appropriate quality factor via $1/Q_{lmpq} = \tan|\epsilon_{lmpq}|$. To keep the analogy between the true lags and the instantaneous lag (53), one may conveniently *define* a quantity Q as the inverse of $\tan|\epsilon|$. This will enable one to express the MacDonald torque as

$$
\tau = \frac{3}{2} G M_{sat}^2 k_2 \frac{R^5}{r(t)^3 r(t - \Delta t)^3} \sin \epsilon + O(i^2/Q)
$$

$$
= \frac{3}{2} G M_{sat}^2 k_2 \frac{R^5}{r^6} Q^{-1} \text{sgn} (\dot{v} - \dot{\theta}) + O(i^2/Q) + O(en/Q^2\chi) + O(Q^{-3}) \quad (54)
$$

Since Q was *defined* as $1/\tan|\epsilon|$, it is not guaranteed to deserve the name of an overall quality factor. At each particular frequency χ_{lmpq}, the corresponding quality factor $Q_{lmpq} \equiv 1/\tan|\epsilon_{lmpq}|$ is related to the peak energy of this mode, $E_{peak}(\chi_{lmpq})$, and to the one-cycle energy loss at this frequency, $\Delta E_{cycle}(\chi_{lmpq})$, via

$$
\Delta E_{cycle}(\chi_{lmpq}) = -\frac{2\pi E_{peak}(\chi_{lmpq})}{Q_{lmpq}}. \quad (55)
$$

However, it is not at all obvious if the quantity Q defined through the longitudinal lag as $Q \equiv 1/\tan|\epsilon|$ interconnects the overall tidal energy with the overall one-cycle loss, in a manner similar to (55). The literature hitherto has always taken for granted that it does. However, the proof (to be presented elsewhere) requires some effort. The proof is based on interpreting $\chi \equiv 2|\dot{v} - \dot{\theta}|$ as an *instantaneous* tidal frequency.

The interconnection between $Q \equiv 1/\tan|\epsilon|$ and the overall energy-damping rate mimics (55) only up to a relative error of order $O(en/Q\chi) = O(en\Delta t)$, i.e., up to an absolute error of order $O(eQ^{-1}n\Delta t)$. This is acceptable, because in realistic settings $n\Delta t \ll 1$.

7.2 Further simplifications available in the zeroth order of $en/Q\chi$

Suppose, we ignore the difference between r and r^*, which are the two locations of the same satellite, separated by the time lag owing to the tidal response. We shall now demonstrate that, though the relative error of this approximations is $O(en/Q\chi)$, after averaging over a satellite period this approximation brings only a $O(e^2n^2/Q^2\chi^2)$ relative error into the expression for the torque.

From the well-known formulae $r = a(1 - e^2)/(1 + e \cos v)$ and $\partial v/\partial M = (1 + e \cos v)^2/(1 - e^2)^{3/2}$ we see that

$$
\Delta r \equiv r(t) - r(t - \Delta t) = -\frac{ae(1 - e^2)}{(1 + e \cos v)^2} \sin v \Delta v + O\left(e(\Delta v)^2\right)
$$

$$
= -\frac{ae \sin v}{(1 - e^2)^{1/2}} n \Delta t + O\left(e(n\Delta t)^2\right).
$$

---

[14]   As the subtended angle is $|(\dot{v} - \dot{\theta})\Delta t|$, its double is equal to the absolute value of ϵ, and not to that of $\epsilon_{220q} = 2(n - \dot{\theta})\Delta t$.

The time lag is interconnected with the phase shift and the quality factor via the relations

$$\chi \Delta t = \epsilon \approx Q^{-1},$$

$\chi = 2|\dot{\theta} - \dot{\nu}|$ being the instantaneous tidal frequency. Hence

$$\Delta r \equiv r(t) - r(t - \Delta t) \approx -a \frac{e}{Q} \frac{n}{\chi} \sin \nu.$$

As Δr is proportional to $\sin \nu$, only terms quadratic in Δr survive averaging. Thus, while in

$$U = -\frac{3}{4} G M_{\text{sat}}^* k_2 \frac{R^5}{r^6} \cos(2\lambda - 2\lambda^* + \epsilon) + O(en/Q\chi) + O(i^2) + O(i^{*2}) + O(ii^*), \tag{56}$$

and

$$\tau = \frac{3}{2} G M_{\text{sat}} M_{\text{sat}}^* k_2 \frac{R^5}{r^6} \sin(2\lambda - 2\lambda^* + \epsilon) + O(en/Q\chi) + O(i^2), \tag{57}$$

the *relative* error is $O(en/Q\chi) + O(i^2)$, in the averaged expression[15]

$$\langle \tau \rangle = -\frac{3 G M_{\text{sat}}^2 k_2}{2R} \left\langle \frac{R^6}{r^6} \sin \epsilon \right\rangle + O(e^2 n^2/Q^3 \chi^2) + O(i^2/Q) \tag{58a}$$

$$= -\frac{3 G M_{\text{sat}}^2 k_2 R}{4\pi a^2} \frac{1}{\left(1 - e^2\right)^{1/2}} \int_0^{2\pi} \frac{R^4}{r^4} \sin \epsilon \, d\nu + O(e^2 n^2/Q^3 \chi^2) + O(i^2/Q). \tag{58b}$$

it is only $O(e^2 n^2/Q^2 \chi^2) + O(i^2)$.

In the above expressions, we asserted after the differentiation that $M_{\text{sat}}^* = M_{\text{sat}}$ and $\lambda^* = \lambda$, implying that the tide-generating and tidally-perturbed moons are one and the same body. As soon as λ is set to be equal to λ^*, the sine function in (58) becomes $\sin \epsilon \approx 1/Q$. So, while the *relative* error in (58) is $O(e^2 n^2/Q^2 \chi^2) + O(i^2)$, the *absolute* error becomes $O(e^2 n^2/Q^3 \chi^2) + O(i^2/Q)$.

The error $O(e^2 n^2/Q^3 \chi^2)$ becomes irrelevant for two reasons. First, our substitution of $\sin \epsilon$ with $\tan \epsilon = 1/Q$ generates a relative error $O(Q^{-2})$, i.e., an absolute error $O(Q^{-3})$.

---

[15] We recall that time averages over one revolution of the satellite about the primary are given by

$$\langle \ldots \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} \ldots d\mathcal{M} = \frac{\left(1 - e^2\right)^{3/2}}{2\pi} \int_0^{2\pi} \ldots \frac{d\nu}{(1 + e \cos \nu)^2},$$

while the planetocentric distance is $r = a \left(1 - e^2\right)/(1 + e \cos \nu)$, with ν being the true anomaly. This way,

$$\left\langle \frac{R^6}{r^6} \sin \epsilon \right\rangle = \frac{\left(1 - e^2\right)^{3/2}}{2\pi} \int_0^{2\pi} \frac{R^6}{r^6} \sin \epsilon \frac{d\nu}{(1 + e \cos \nu)^2}$$

$$= \frac{\left(1 - e^2\right)^{3/2}}{2\pi} \int_0^{2\pi} \frac{R^6}{r^6} \sin \epsilon \frac{r^2 d\nu}{a^2 \left(1 - e^2\right)^2}$$

$$= \frac{R^2}{a^2} \frac{1}{2\pi \left(1 - e^2\right)^{1/2}} \int_0^{2\pi} \frac{R^4}{r^4} \sin \epsilon \, d\nu.$$

Second, as explained in the end of Subsect. 7.1, the uncertainties inherent in our definition of the overall quality factor Q entail an absolute error $O(en/Q^2\chi)$. Each of these two errors exceeds $O(e^2n^2/Q^3\chi^2)$. We can then write:

$$\langle\tau\rangle = -\frac{3GM_{\text{sat}}^2 k_2}{2R}\left\langle\frac{\text{sgn}\,(\dot\theta - \dot\nu)}{Q}\frac{R^6}{r^6}\right\rangle + O(Q^{-3}) + O(i^2/Q) + O(en/Q^2\chi) \quad (59a)$$

$$= -\frac{3GM_{\text{sat}}^2 k_2 R}{4\pi a^2}\frac{1}{(1-e^2)^{1/2}}\int_0^{2\pi}\frac{R^4}{r^4}\frac{\text{sgn}\,(\dot\theta - \dot\nu)}{Q}d\nu$$

$$+ O(Q^{-3}) + O(i^2/Q) + O(en/Q^2\chi). \quad (59b)$$

8 Use and abuse of approximation (56–59)

Just as with the formula (56) for the potential, the elegant expression (57) for the torque remains correct only to the zeroth order in e/Q, while (59) is valid to the first order. This is the reason why the convenience of this approximation and of its corollaria is somewhat deceptive. Nevertheless, the (56)–(59) were employed by many an author.

Goldreich and Peale (1996) used them to build a theory containing terms up to e^7. Now we see that some coefficients in their theory of capture into resonances must be reconsidered. The same pertains to some coefficients in the theory of Mercury's rotation, recently offered by Peale (2005). Fortunately, the key conclusions of Peale (2005) stay unaltered, despite the corrections needed in the said coefficients.[16]

Interestingly, Kaula (1968) fell into this temptation, and so did Goldreich (1966b). Equation (4.5.29) in Kaula (1968), as well as Eq. (15) in Goldreich (1966b), is but the above formula (59) with the inverse quality factor taken out of the integral:

$$\tau^{Kaula} = -\frac{3GM_{\text{sat}}^2 k_2 R}{4\pi a^2 Q}\frac{1}{(1-e^2)^{1/2}}\int_0^{2\pi}\frac{R^4}{r^4}\text{sgn}\,(\dot\theta - \dot\nu)d\nu.$$

(Kaula 1968, Eq. 4.5.29)

Besides the afore-mentioned fact that this approach contains a relative error $O(en/Q\chi) + O(Q^{-2}) + O(i^2)$, it suffers a greater defect. Taking Q^{-1} out of the integral is illegitimate, because it implies frequency-independence of Q. This is then incompatible with Kaula's implicit assumption of a constant Δt, an assumption tacitly present in (42).[17]

Goldreich (1966b) and Kaula (1968) used this oversimplified formula to investigate librations of a satellite trapped in a 1:1 resonance. Other authors used it to evaluate despinning rates of bodies outside this resonance. We shall dwell on the latter case in Sect. 10.

9 Can the quality factor scale as a positive power of the tidal frequency?

As of now, the functional form of the dependence $Q(\chi)$ for Jovian planets remains unknown. For terrestrial planets, the model $Q \sim 1/\chi$ is definitely incompatible with the geophysical data. A convincing volume of measurements firmly witnesses that Q of the mantle scales as the tidal frequency to a *positive* fractional power:

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[16] Stan Peale, private communication, 2007.

[17] This would be incompatible with the MacDonald (1964) treatment as well, because MacDonald's formalism necessitates the rheology $Q \sim 1/\chi$. We shall return to this point in Subsect. 10.1.

$$Q = (\mathcal{E}\chi)^{\alpha}, \quad \text{where } \alpha = 0.3 \pm 0.1, \tag{60}$$

$\mathcal{E}$ being an integral rheological parameter with dimensions of time. This rheology is incompatible with the postulate of frequency-independent time-delay. Therefore, insertion of the realistic model (60) into the formula presented in Sect. 8 will remain insufficient. An honest calculation should be based on averaging the Darwin–Kaula–Goldreich formula (38), with the actual scaling law (60) inserted therein, and with the appropriate dependence $\Delta t_{\mathrm{lmpq}}(\chi_{\mathrm{lmpq}})$ taken into account (see formula (83) below).

9.1 The "paradox"

Although among geophysicists the scaling law (60) has long become common knowledge, in the astronomical community it is often met with prejudice. The prejudice stems from the fact that, in the expression for the torque, Q stands in the denominator:

$$\tau \sim \frac{1}{Q}. \tag{61}$$

At the instant of crossing the synchronous orbit, the principal tidal frequency χ_{2200} becomes nil, for which reason insertion of

$$Q \sim \chi^{\alpha}, \quad \alpha > 0 \tag{62}$$

into (61) seems to entail an infinitely large torque at the instant of crossing:

$$\tau \sim \frac{1}{Q} \sim \frac{1}{\chi^{\alpha}} \to \infty, \quad \chi \to 0, \tag{63}$$

a clearly unphysical result.

Another, very similar objection to (60) originates from the fact that the quality factor is inversely proportional to the phase shift: $Q \sim 1/\epsilon$. As the shift (24) vanishes on crossing the synchronous orbit, one may think that the value of the quality factor must, effectively, approach infinity. On the other hand, the principal tidal frequency vanishes on crossing the synchronous orbit, for which reason (60) makes the quality factor vanish. Thus, we come to a contradiction.

For these reasons, the long-entrenched opinion is that these models introduce discontinuities into the equations and can thus be considered as unrealistic approximations for rotating bodies.

It is indeed true that, while law (60) works over scales shorter than the Maxwell time (about 10^2 years for most minerals), it remains subject to discussion in regard to longer timescales. Nonetheless, it should be clearly emphasised that the infinities emerging at the synchronous-orbit crossing can in no way disprove any kind of rheological model. They can only disprove the flawed mathematics whence they provence.

9.2 A case for reasonable doubt

To evaluate the physical merit of the alleged infinite-torque "paradox", recall the definition of the quality factor. As part and parcel of the linearity approximation, the overall damping inside a body is expanded in a sum of attenuation rates corresponding to each periodic disturbance:

$$\langle \dot{E} \rangle = \sum_i \langle \dot{E}(\chi_i) \rangle \tag{64}$$

where, at each frequency χ_i,

$$\langle \dot{E}(\chi_i) \rangle = -2\chi_i \frac{\langle E(\chi_i) \rangle}{Q(\chi_i)} = -\chi_i \frac{E_{\text{peak}}(\chi_i)}{Q(\chi_i)}, \tag{65}$$

$\langle \ldots \rangle$ designating an average over a flexure cycle, $E(\chi_i)$ denoting the energy of deformation at the frequency χ_i, and $Q(\chi_i)$ being the quality factor of the medium at this frequency.

This definition by itself leaves enough room for doubt in the above "paradox". As can be seen from (65), the dissipation rate is proportional not to $1/Q(\chi)$ but to $\chi/Q(\chi)$. This way, for the dependence $Q \sim \chi^\alpha$, the dissipation rate $\langle \dot{E} \rangle$ will behave as $\chi^{1-\alpha}$. In the limit of $\chi \to 0$, this scaling law portends no visible difficulties, at least for the values of α up to unity. While raising α above unity may indeed be problematic, there seem to be no fundamental obstacle to having materials with positive α taking values up to unity. So far, such values of α have caused no paradoxes, and there seems to be no reason for any infinities to show up.

9.3 The phase shift and the quality factor

As another preparatory step, we recall that, rigorously speaking, the torque is proportional not to the phase shift ϵ itself but to $\sin\epsilon$. From (37) and (60), we obtain:

$$|\sin\epsilon| = \frac{1}{\sqrt{1 + Q^2}} = \frac{1}{\sqrt{1 + \mathcal{E}^{2\alpha}\chi^{2\alpha}}}. \tag{66}$$

We see that only for large values of Q one can approximate $|\sin\epsilon|$ with $1/Q$ (crossing of the synchronous orbit *not* being the case). Generally, in any expression for the torque, the factor $1/Q$ must always be replaced with $1/\sqrt{1 + Q^2}$. Thus, instead of (61) we must write:

$$\tau \sim |\sin\epsilon| = \frac{1}{\sqrt{1 + Q^2}} = \frac{1}{\sqrt{1 + \mathcal{E}^{2\alpha}\chi^{2\alpha}}}, \tag{67}$$

$\mathcal{E}$ being a dimensional constant from (60).

Although this immediately spares us from the fake infinities at $\chi \to 0$, we still are facing a strange situation: it follows from (66) that, for a positive α and vanishing χ, the phase lag ϵ must be approaching $\pi/2$, thereby inflating the torque to its maximal value (while on physical grounds the torque should vanish for zero χ). Evidently, some important details are still missing from the picture.

9.4 The stone rejected by the builders

To find the missing link, recall that Kaula (1964) described tidal damping by employing the method suggested by Darwin (1880): he accounted for attenuation by merely adding a phase shift to every harmonic involved—an empirical approach intended to make up for the lack of a consistent hydrodynamical treatment with viscosity included. It should be said, however, that, prior to the work of 1880, Darwin had published a less known article (Darwin 1879), in which he attempted to construct a self-consistent theory, one based on the viscosity factor of the mantle, and not on empirical phase shifts inserted by hand. Darwin's conclusions of 1879 were summarised and explained in a more general mathematical setting by Alexander (1973).

The pivotal result of the self-consistent hydrodynamical study is the following. When a variation of the potential of a tidally distorted planet, $U(\vec{r})$, is expanded over the Legendre functions $P_{lm}(\sin\phi)$, each term of this expansion will acquire not only a phase lag but also a factor describing a change in amplitude. This forgotten factor, derived by Darwin (1879), is nothing else but $\cos\epsilon$. Its emergence should in no way be surprising if we recall that the damped, forced harmonic oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega_o^2 x = F e^{i\lambda t} \tag{68}$$

evolves as

$$x(t) = C_1 e^{\left(-\gamma + i\sqrt{\omega_o^2 - \lambda^2}\right)t} + C_2 e^{\left(-\gamma - i\sqrt{\omega_o^2 - \lambda^2}\right)t} + \frac{F\cos\epsilon}{\omega_o^2 - \lambda^2} e^{i(\lambda t - \epsilon)}, \tag{69}$$

where the phase lag is

$$\tan\epsilon = \frac{2\gamma\lambda}{(\omega_o^2 - \lambda^2)} \tag{70}$$

and the first two terms in (69) are damped away in time.[18]

In the works by Darwin's successors, the allegedly irrelevant factor of $\cos\epsilon$ fell through the cracks, because the lag was always asserted to be small. In reality, though, each term in the Fourier expansions (21), (30–35), and (38) should be amended with $\cos\epsilon_{\text{lmpq}}$. Likewise, the correct versions of (50–51) and (54) should contain an extra factor of $\cos\epsilon_{2200}$. For the same reason, instead of (67), we should write down:

$$\tau \sim |\cos\epsilon \sin\epsilon| = \frac{Q}{\sqrt{1 + Q^2}} \frac{1}{\sqrt{1 + Q^2}} = \frac{\mathcal{E}^\alpha \chi^\alpha}{1 + \mathcal{E}^{2\alpha} \chi^{2\alpha}}. \tag{71}$$

At this point, it would be tempting to conclude that, since (71) vanishes in the limit of $\chi \to 0$, *for any sign of α*, then no paradoxes happen on the satellite's crossing the synchronous orbit. Sadly, this straightforward logic would be too simplistic.

In fact, prior to saying that $\cos\epsilon\sin\epsilon \to 0$, we must take into consideration one more subtlety missed so far. As demonstrated in the Appendix, taking the limit of $Q \to 0$ is a nontrivial procedure, because at small values of Q the interconnection between the lag and the Q factor becomes very different from the conventional $Q = \cot|\epsilon|$. A laborious calculation shows that, for $Q < 1 - \pi/4$, the relation becomes:

$$|\sin\epsilon\cos\epsilon| = (3Q)^{1/3}\left[1 - \frac{4}{5}(3Q)^{2/3} + O(Q^{4/3})\right],$$

which indeed vanishes for $Q \to 0$. Both ϵ_{2200} and the appropriate component of the torque change their sign on the satellite crossing the synchronous orbit.

So the main conclusion remains in force: nothing wrong happens on crossing the synchronous orbit, Q.E.D.

---

[18] As demonstrated by Alexander (1973), this example indeed has relevance to the hydrodynamical theory of Darwin, and is not a mere illustration. Alexander (1973) also explained that the emergence of the $\cos\epsilon$ factor is generic. (Darwin (1879)) had obtained it in the simple case of $l = 2$ and for a special value of the Love number: $k_2 = 1.5$.)

A further investigation of this issue was undertaken in a comprehensive work by Churkin (1998), which unfortunately has never been published in English because of a tragic death of its Author. In this preprint, Churkin explored the frequency-dependence of both the Love number k_2 and the quality factor within a broad variety of rheological models, including those of Maxwell and Voight. It follows from Churkin's formulae that within the Voight model the dynamical k_2 relates to the static one as $\cos\epsilon$. In the Maxwell and other models, the ratio approaches $\cos\epsilon$ in the low-frequency limit.

10 Tidal despinning

The following formula for the average deceleration rate $\ddot{\theta}$ of a planet due to a tide-raising satellite has often appeared in the literature:

$$\langle \ddot{\theta} \rangle = -\mathcal{K} \left[\dot{\theta} \mathcal{A}(e) - n \mathcal{N}(e) \right], \tag{72}$$

where

$$\mathcal{A}(e) = \left(1 + 3e^2 + \frac{3}{8} e^4 \right) \left(1 - e^2 \right)^{-9/2}, \tag{73}$$

and

$$\mathcal{N}(e) = \left(1 + \frac{15}{2} e^2 + \frac{45}{8} e^4 + \frac{5}{16} e^6 \right) \left(1 - e^2 \right)^{-6}, \tag{74}$$

θ being the sidereal angle, $\dot{\theta}$ being the primary's spin rate, $\mathcal{K}$ being some constant, and the angular brackets designating an average over one revolution of the secondary about the primary. This expression was derived by different methods in Goldreich and Peale (1996) and Hut (1981), and was later employed by Dobrovolskis (1995, 2007) and Correia and Laskar (2004).[19]

10.1 Derivation by means of the MacDonald torque

The following proof of (72–74) is implied in Goldreich and Peale (1996) and is presented in more detail in Dobrovolskis (2007). Their starting point was the MacDonald torque (59). Hut (1981), who approached the issue in the language of the Lagrange-type planetary equations, took into account, in the disturbing function, only the leading term of series (21), and substituted the principal tidal frequency $\chi_{2200} = 2|\dot{\theta} - n|$ with the synodal frequency $\chi = 2|\dot{\theta} - \dot{\nu}|$. Thereby, his approach was equivalent to that of Goldreich and Peale (1996) and Dobrovolskis (2007).

Although not necessarily assumed by these authors,[20] their method, as we saw in the section above, inherently implied the following assertions:

- (I)   The quantity $\chi = 2|\dot{\theta} - \dot{\nu}|$ is treated as an instantaneous tidal frequency. Accordingly, the overall quality factor Q is implied to be a function not of the principal frequency χ_{2200} but of the instantaneous frequency χ.
- (II) The functional form of this dependence is chosen as $Q = (\Delta t)^{-1} \chi^{-1}$, where Δt is the time lag.
- (III) The time lag Δt is frequency-independent. This assertion is equivalent to (II), as can be demonstrated from (24).

---

[19] Our formula (78) differs from formula (4) in Correia and Laskar (2004) by a factor of n/χ, because in *Ibid.* the quality factor had been introduced as $1/(n\Delta t)$ and not as $1/(\chi \Delta t)$.

[20] It should be mentioned that the original treatment by MacDonald (1964) is inherently contradictory. On the one hand, MacDonald postulates (following Gerstenkorn 1955) that there exists one overall double bulge. As explained in Subsect. 7.1 above, this assertion unavoidably implies constancy of the time lag Δt, so that $Q \sim 1/\chi$ and $\epsilon \sim \chi$. However, MacDonald (1964) erroneously set Q (and, thence, also ϵ) frequency-independent, an assertion incompatible with his and Gerstenkorn's postulate of existence of an overall double bulge.

Whenever in the current paper we refer to MacDonald's torque, we always imply his postulate that one double bulge exists. At the same time, to make the MacDonald–Gerstenkorn treatment consistent, we always adjust the MacDonald–Gerstenkorn treatment by letting Q and ϵ scale as $1/\chi$ and χ, correspondingly.

Beside this, those authors neglected the order-$en/Q\chi$ difference between r and r^* in (54), generating a relative error in τ of order $O(en/Q\chi)$ (which, luckily, reduced to $O(e^2 n^2/Q^2\chi^2)$ after orbital averaging). They also substituted $\sin\epsilon$ with $1/Q$, causing a relative error of order $O(1/Q^2)$, because in reality Q is the reciprocal of $\tan\epsilon$, not of $\sin\epsilon$.

Assertion (II) can be written down in more generic notation:

$$Q = (\mathcal{E}\chi)^{\alpha}, \quad \text{with} \quad \alpha = -1. \tag{75}$$

This form of the scaling law is more convenient, for it leaves one an opportunity to switch to different values of α. For any value of α (not only for -1), the constant $\mathcal{E}$ is an integral rheological parameter (with the dimension of time), whose physical meaning is explained in Efroimsky and Lainey (2007). It can be shown that in the particular case of $\alpha = -1$ the parameter $\mathcal{E}$ coincides with Δt. In realistic situations, α differs from -1, while the parameter $\mathcal{E}$ is related to the time lag in a more sophisticated way (*Ibid.*).

To show how (72–74) stem from the above Assertions, keep for the time being $\alpha = -1$. Also recall that the torque is despinning (so $\dot\theta > n$), and that for the averages over time

$$\langle\ddot\theta\rangle = \frac{\langle\tau\rangle}{C}, \tag{76}$$

$C = \xi M_{\text{planet}} R^2$ being the maximal moment of inertia of the planet. (For a homogeneous spherical planet, $\xi = 2/5$.) Then plug (75) into (59) and average the torque:[21]

$$\langle\tau\rangle = -\frac{3GM_{\text{sat}}^2 k_2 \mathcal{E}}{R}\left\langle (\dot\theta - \dot\nu)\frac{R^6}{r^6}\right\rangle + O(i^2/Q) + O(Q^{-3}) + O(en/Q^2\chi)$$

$$= -\frac{3GM_{\text{sat}}^2 k_2 \mathcal{E}}{R}\dot\theta\left\langle \frac{R^6}{r^6}\right\rangle + \frac{3GM_{\text{sat}}^2 k_2 \mathcal{E}}{R}\left\langle \dot\nu\frac{R^6}{r^6}\right\rangle$$

$$+ O(i^2/Q) + O(Q^{-3}) + O(en/Q^2\chi) \tag{77a}$$

$$= -\frac{3GM_{\text{sat}}^2 k_2 \mathcal{E}}{R}\dot\theta\frac{R^6}{a^6}\left(1 - e^2\right)^{-9/2}\frac{1}{2\pi}\int_0^{2\pi}(1 + e\cos\nu)^4\,d\nu$$

$$+ \frac{3GM_{\text{sat}}^2 k_2 \mathcal{E}}{R}n\frac{R^6}{a^6}\left(1 - e^2\right)^{-6}\frac{1}{2\pi}\int_0^{2\pi}(1 + e\cos\nu)^6\,d\nu$$

$$+ O(i^2/Q) + O(Q^{-3}) + O(en/Q^2\chi), \tag{77b}$$

where the absolute error $O(en/Q^2\chi)$ emerges due to an uncertainty in the definition of the overall quality factor Q employed in MacDonald's model.

Evaluation of the above integrals is trivial and indeed leads to (72–74), the constant being

$$\mathcal{K} = \frac{3GM_{\text{sat}}^2 k_2 \mathcal{E}}{CR}\frac{R^6}{a^6} = \frac{3n^2 M_{\text{sat}}^2 k_2 \Delta t}{\xi M_{\text{planet}}(M_{\text{planet}} + M_{\text{sat}})}\frac{R^3}{a^3}$$

$$= \frac{3n M_{\text{sat}}^2 k_2}{\xi Q M_{\text{planet}}(M_{\text{planet}} + M_{\text{sat}})}\frac{R^3}{a^3}\frac{n}{\chi}, \tag{78}$$

where we used the fact that for $\alpha = -1$ the rheological parameter $\mathcal{E}$ is simply the lag Δt.

---

[21] As explained in the paragraph preceding formula (59), substitution of $\sin\epsilon$ with $1/Q$ in the expression for torque generates a relative error $O(Q^{-2})$, i.e., an absolute error $O(Q^{-3})$. Instead of inserting (75) into (59), one may directly use (53). Still, approximation of $\sin\epsilon$ with ϵ will entail, in (77) and its corollaria, a relative error $O(Q^{-2})$ and an absolute error $O(Q^{-3})$. The situation will become more complicated in the special case of low values of the quality factor. See the Appendix below.

It should also be added that, since (77b) contains a relative error $O(Q^{-2})$, the usefulness of the e^4 and e^6 terms in (73–74) depends on the values of the eccentricity and the quality factor. If, for example, $Q = 70$, then the e^4 terms become unimportant for $e < 0.12$, while the e^6 terms become unimportant for $e < 0.24$.

To draw to a close, we would mention that besides the above formula (72), in the literature hitherto we saw its sibling, an expression derived in a similar way, but with Assertion II rejected in favor of treating Q as a frequency-independent constant. The result of this treatment suffers an incurable birth trauma—the incompatibility between the frequency-independence of Δt and the frequency-independence of Q.

10.2 Calculation based on the Darwin torque

The following alternative derivation is based on the same Assertions (I–III) and, naturally, leads to the same results. The idea is to calculate the despinning rate not in terms of the Mac-Donald torque, but in terms of the Darwin torque, keeping the eccentricity-caused relative error at the level of $O(e^6)$.

To keep the inclination-caused relative error at the level of $O(i^2)$, we still assume, in (34), that $l = 2$, $m = 2$, $p = 0$. As for the the values of q, we keep only the ones giving us terms of order up to e^4, inclusively. Besides, we assume the phase lags to be small, so that $\sin \epsilon_{lmpq} = \epsilon_{lmpq} + O(\epsilon^3) = \epsilon_{lmpq} + O(Q^{-3})$. Under all these presumptions, the constant part of the tidal torque can be approximated with

$$\tau_{l=2} = \frac{3}{2} GM_{\text{sat}}^2 R^5 a^{-6} k_2 \sum_{q=-2}^{2} G_{20q}^2 \sin \epsilon_{220q} + O(e^6/Q) + O(i^2/Q) \tag{79a}$$

$$= \frac{3}{2} GM_{\text{sat}}^2 R^5 a^{-6} k_2 \sum_{q=-2}^{2} G_{20q}^2 \epsilon_{220q} + O(e^6/Q) + O(i^2/Q) + O(Q^{-3}), \tag{79b}$$

where, according to the tables (Kaula 1966),

$$G_{20-2}^2 = 0, \quad G_{20-1}^2 = \frac{e^2}{4} - \frac{e^4}{16} + O(e^6), \quad G_{200}^2 = 1 - 5e^2 + \frac{63}{8} e^4 + O(e^6),$$

$$G_{201}^2 = \frac{49}{4} e^2 - \frac{861}{16} e^4 + O(e^6), \quad G_{202}^2 = \frac{289}{4} e^4 + O(e^6), \tag{80}$$

and, according to formula (24),

$$\epsilon_{220-2} = (-2\dot{\theta}) \Delta t_{220-2}, \quad \epsilon_{220-1} = (-2\dot{\theta} + n) \Delta t_{220-1}$$

$$\epsilon_{2200} = (-2\dot{\theta} + 2n) \Delta t_{2200}, \quad \epsilon_{2201} = (-2\dot{\theta} + 3n) \Delta t_{2201},$$

$$\epsilon_{2202} = (-2\dot{\theta} + 4n) \Delta t_{2202}. \tag{81}$$

Provided the quality factor scales as inverse frequency, all the time lags are the same constant Δt, so the above formulae all together entail, in the case of nonresonant prograde spin:

$$\ddot{\theta} = \frac{\tau}{C} = \mathcal{K} \left[-\dot{\theta} \left(1 + \frac{15}{2} e^2 + \frac{105}{4} e^4 + O(e^6) \right) \right.$$

$$\left. + n \left(1 + \frac{27}{2} e^2 + \frac{573}{8} e^4 + O(e^6) \right) \right] + O(i^2/Q) + O(Q^{-3}), \tag{82}$$

which coincides with (72–74) to the order e^4, inclusively, provided we substitute χ_{2200} instead of χ in the expression (78) for $\mathcal{K}$.

10.3 Rheologies different from $Q \sim 1/\chi$

A part and parcel of both afore-presented methods was the assertion of all the time lags Δt_{lmpq} being equal. In reality, the time lags vary from one harmonic to another.

Any particular functional form of the dependence $\Delta t(\chi)$ fixes the rheology: for example, the frequency-independence of Δt constrains the value of the exponential α to -1 (while the parameter $\mathcal{E}$ becomes simply Δt). However, for an arbitrary $\alpha \neq -1$ the lags will read (Efroimsky and Lainey 2007):

$$\Delta t_{lmpq} = \mathcal{E}^{-\alpha} \chi_{lmpq}^{-(\alpha+1)} \tag{83}$$

While the MacDonald approach cannot be generalised to $\alpha \neq -1$, the Darwin–Kaula–Goldreich method can be well combined with (83). To this end, we shall insert (80–81) and (83) into (79a), and shall also employ the evident formulae

$$\cos \epsilon_{lmpq} = \frac{|\cot \epsilon_{lmpq}|}{\sqrt{1 + \cot^2 \epsilon_{lmpq}}} = \frac{Q_{lmpq}}{\sqrt{1 + Q_{lmpq}^2}} = \frac{\mathcal{E}^\alpha \chi_{lmpq}^\alpha}{\sqrt{1 + \mathcal{E}^{2\alpha} \chi_{lmpq}^{2\alpha}}}, \tag{84}$$

$$\sin \epsilon_{lmpq} = \sin |\epsilon_{lmpq}| \, \mathrm{sgn}\, \omega_{lmpq} = \frac{\mathrm{sgn}\, \omega_{lmpq}}{\sqrt{1 + \cot^2 \epsilon_{lmpq}}} = \frac{\mathrm{sgn}\, \omega_{lmpq}}{\sqrt{1 + Q_{lmpq}^2}}$$

$$= \frac{\mathrm{sgn}\, \omega_{lmpq}}{\sqrt{1 + \mathcal{E}^{2\alpha} \chi_{lmpq}^{2\alpha}}}, \tag{85}$$

with ω_{lmpq} given by (25), and $|\epsilon_{lmpq}|$ assumed (for reasons explained in the Appendix) not to approach too close to $\pi/2$. This will give us the following expression for (the constant part of) the deceleration rate of a non-resonant prograde spin:

$$\ddot\theta = \frac{\tau}{C} = -\frac{3}{2} \frac{GM_{\mathrm{sat}}^2}{a^3} \frac{R^5}{a^3} \frac{k_2}{\xi M_{\mathrm{planet}} R^2} \left[\frac{e^2}{4} \mathrm{sgn}\, (2\dot\theta - n) \frac{\mathcal{E}^\alpha |2\dot\theta - n|^\alpha}{1 + \mathcal{E}^{2\alpha} |2\dot\theta - n|^{2\alpha}} \right.$$

$$+ \left(1 - 5e^2 + \frac{63}{8} e^4\right) \mathrm{sgn}\, (2\dot\theta - 2n) \frac{\mathcal{E}^\alpha |2\dot\theta - 2n|^\alpha}{1 + \mathcal{E}^{2\alpha} |2\dot\theta - 2n|^{2\alpha}}$$

$$+ \left(\frac{49}{4} e^2 - \frac{861}{16} e^4\right) \mathrm{sgn}\, (2\dot\theta - 3n) \frac{\mathcal{E}^\alpha |2\dot\theta - 3n|^\alpha}{1 + \mathcal{E}^{2\alpha} |2\dot\theta - 3n|^{2\alpha}}$$

$$+ \left. \frac{289}{4} e^4 \mathrm{sgn}\, (2\dot\theta - 4n) \frac{\mathcal{E}^\alpha |2\dot\theta - 4n|^\alpha}{1 + \mathcal{E}^{2\alpha} |2\dot\theta - 4n|^{2\alpha}} \right]$$

$$+ O(i^2/Q) + O(e^6/Q). \tag{86}$$

Be mindful, that a naive substitution of the formula (85) for $\sin \epsilon_{lmpq}$ into (79a) would result in an expression for the torque, attaining its maxima on approach to resonances (for a positive α), an evidently unphysical behaviour. As explained in Sect. 9, there exists a profound physical reason, for which the actual multiplier in (79a) must be not $\sin \epsilon_{lmpq}$ but: $\sin \epsilon_{lmpq} \cos \epsilon_{lmpq}$. Mathematically, the presence of the cosine is irrelevant unless χ_{lmpq} and Q_{lmpq} approach zero.

If, however, χ_{lmpq} becomes very small (i.e., if we approach a resonance), it is this long-omitted (though known yet to Darwin 1879) cosine multiplier that saves us from the unphysical maxima—See Sect. 9 above.

Under the extra assumptions[22] of $|\epsilon_{\text{lmpq}}| \ll 1$ and $n \ll \dot{\theta}$, formula (90) simplifies to

$$\ddot{\theta} = \frac{\tau}{C} = \mathcal{K}\left[-\dot{\theta}\left(1 + \frac{15}{2}e^2 + \frac{105}{4}e^4 + O(e^6)\right)\right.$$

$$+ n\left(1 + \left(\frac{15}{2} - 6\alpha\right)e^2 + \left(\frac{105}{4} - \frac{363}{8}\alpha\right)e^4 + O(e^6)\right)\right]$$

$$+ O(i^2/Q) + O(Q^{-3}) + O(\alpha e^2 n/\dot{\theta}) \tag{87}$$

$$\approx \mathcal{K}\left[-\dot{\theta}\left(1 + \frac{15}{2}e^2\right) + n\left(1 + \left(\frac{15}{2} - 6\alpha\right)e^2\right)\right], \tag{88}$$

where the overall factor $\mathcal{K}$ is given by

$$\mathcal{K} = \frac{3n^2 M_{\text{sat}}^2 k_2 \Delta t_{2200}}{\xi M_{\text{planet}}(M_{\text{planet}} + M_{\text{sat}})} \frac{R^3}{a^3}$$

$$= \frac{3n M_{\text{sat}}^2 k_2}{\xi Q_{2200} M_{\text{planet}}(M_{\text{planet}} + M_{\text{sat}})} \frac{R^3}{a^3} \frac{n}{\chi_{2200}}, \tag{89}$$

an expression identical to (78), except that it contains Δt_{2200}, Q_{2200}, and χ_{2200} instead of Δt, Q, and χ, correspondingly.

Were α equal to -1, sum (87) would coincide with (82), provided $\dot{\theta} > 2n$ (but not otherwise!). For realistic mantles and crusts, though, the values of α will, as pointed above, reside within the interval $0.2 - 0.4$ (closer to 0.2 for partial melts).

11 Conclusions

In the article, thus far, we have provided a detailed review of a narrow range of topics. Our goal was to punctiliously spell out the assumptions that often remain implicit, and to bring to light those steps in calculations, which are often omitted as "self-evident".

This has helped us to demonstrate that MacDonald-style formula (57) for the tidal torque is valid only in the zeroth order of $en/Q\chi$, while its time-average is valid only in the first

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[22] The smallness of $|\epsilon_{\text{lmpq}}|$ enables one to employ (79b) instead of (79a). Then the multipliers

$\dfrac{\text{sgn}\,\omega_{\text{lmpq}}\,\mathcal{E}^\alpha\,\chi_{\text{lmpq}}^\alpha}{1+\mathcal{E}^{2\alpha}\chi_{\text{lmpq}}^{2\alpha}}$ in (90) become $\epsilon_{\text{lmpq}} = \omega_{\text{lmpq}}\Delta t_{2200}\dfrac{\Delta t_{\text{lmpq}}}{\Delta t_{2200}} = \chi_{\text{lmpq}}\Delta t_{2200}\,\text{sgn}\,\omega_{\text{lmpq}}\left(\dfrac{\chi_{2200}}{\chi_{\text{lmpq}}}\right)^{\alpha+1}$

$= \chi_{2200}\Delta t_{2200}\,\text{sgn}\,\omega_{\text{lmpq}}\left(\dfrac{\chi_{2200}}{\chi_{\text{lmpq}}}\right)^\alpha = \chi_{2200}\Delta t_{2200}\,\text{sgn}\,\omega_{\text{lmpq}}\left(1 + \dfrac{\chi_{\text{lmpq}}-\chi_{2200}}{\chi_{2200}}\right)^{-\alpha}$. Specifically,

$\epsilon_{220q} = \text{sgn}\,\omega_{220q}\Delta t_{2200}\chi_{2200}\left(1 + \dfrac{\chi_{220q}-\chi_{2200}}{\chi_{2200}}\right)^{-\alpha} \approx \text{sgn}\,\omega_{220q}\Delta t_{2200}\chi_{2200}\left(1 - \alpha\dfrac{\chi_{220q}-\chi_{2200}}{\chi_{2200}}\right)$

$= \text{sgn}\,\omega_{220q}\Delta t_{2200}\left[(1+\alpha)\chi_{2200}\alpha\chi_{220q}\right]$, the latter approximation being legitimate only under

the condition of $\chi_{220q} - \chi_{2200} \ll \chi_{2200}$, which turns out to be equivalent to $n \ll \dot{\theta}$. For example,

$\dfrac{\chi_{2202}-\chi_{2200}}{\chi_{2200}} = \dfrac{(-2\dot{\theta}+4n)-(-2\dot{\theta}+2n)}{-2\dot{\theta}+2n} = \dfrac{2n}{-2\dot{\theta}+2n}$. So approximating ϵ_{220q}, for $q = -2, -1, 0, 1, 2$,

we arrive at formula (87).

order. These restrictions mean that in the popular expressions for tidal despinning rate the terms with higher powers of e become significant only for large eccentricities. Their significance is limited even further by the error $O(Q^{-3})$ emerging when the sine of the phase lag gets approximated by the inverse quality factor—see formula (59) and the paragraph preceding it.

We have demonstrated that in the case, when the inclinations are small and the phase lags of the tidal harmonics are proportional to the frequency, the Darwin–Kaula expansion is equivalent to a corrected version of the MacDonald formalism. The latter method rests on the assumption of existence of one total double bulge. The necessary correction to MacDonald's approach would be to assert (following Singer (1968)) that the phase lag of this integral bulge is not constant, but is proportional to the instantaneous synodal frequency $2(\dot{v} - \dot{\theta})$, where v and θ are the true anomaly and the sidereal angle. Any rheology different from this one will violate the equivalence of the Darwin–Kaula and MacDonald descriptions. It remains unexplored if their equivalence is violated also by setting the inclination high.

We have demonstrated that no "paradoxes" ensue from the frequency-dependence $Q \sim \chi^{\alpha}$, with $\alpha = 0.3 \pm 0.1$, found for the mantle.

We have investigated the limitations of the popular formula interconnecting the quality factor Q and the phase lag ϵ. It turns out that for low values of the quality factor (much less than 10), the customary formula $Q = \cot|\epsilon|$ should be substituted with a much more complicated relation.

Finally, we examined two derivations of the popular expressions (72–74), and have pointed out that these expressions have limitations related to the frequency-dependence of the quality factor. First, dependent upon the values of e and Q, the high-order terms in these expressions may become significant only for large eccentricities. Second, the expansion of the deceleration rate in even powers of e will be different if Δt is frequency-dependent (which is the case for solid materials). These two circumstances do not necessarily disprove any major result achieved in the bodily-tide theory. However, some coefficients may now have to be reconsidered.

For the realistic rheology of terrestrial bodies, the despinning rate, in the absence of tidal locking, is given by our formulae (86–89).

Appendix. The lag and the quality factor: is the formula $Q = \cot|\epsilon|$ universal?

The interrelation between the quality factor Q and the phase lag ϵ is long-known to be

$$Q = \cot|\epsilon|, \qquad (90)$$

and its derivation can be found in many books. In Appendix A2 of Efroimsky and Lainey (2007), that derivation is reproduced, with several details that are normally omitted in the literature. Among other things, we pointed out that the interrelation has exactly the form (90)

only in the limit of small lags. For large phase lags, the form of this relation will change considerably.

Since in Sect. 9 of the current paper we address the case of large lags, it would be worth reconsidering the derivation presented in Efroimsky and Lainey (2007), and correcting a subtle omission made there. Before writing formulae, let us recall that, at each frequency χ in the spectrum of the deformation, the quality factor (divided by 2π) is defined as the peak energy stored in the system divided by the energy damped over a cycle of flexure:

$$Q(\chi) \equiv -\frac{2\pi\,E_{\text{peak}}(\chi)}{\Delta E_{\text{cycle}}(\chi)}, \tag{91}$$

where $\Delta E_{\text{cycle}}(\chi) < 0$ as we are talking about energy losses.[23]

An attempt to consider large lags (all the way up to $|\epsilon| = \pi/2$) sets the values of $Q/2\pi$ below unity. As the dissipated energy cannot exceed the energy stored in a free oscillator, the question becomes whether the values of $Q/2\pi$ can be that small. To understand that they can, recall that in this situation we are considering an oscillator, which is not free but is driven (and is overdamped). The quality factor being much less than unity simply implies that the eigenfrequencies get damped away during less than one oscillation. Nonetheless, motion goes on due to the driving force.

Now let us switch to the specific context of tides. To begin with, let us recall that the dissipation rate in a tidally distorted primary is well approximated by the work that the secondary carries out to deform the primary:

$$\dot{E} = -\int \rho\vec{V}\cdot\nabla W \, d^3x \tag{92}$$

ρ, $\vec{V}$, and W denoting the density, velocity, and tidal potential inside the primary. The expression on the right-hand side can be transformed by means of the formula

$$\begin{aligned}
\rho\,\vec{V}\cdot\nabla W &= \nabla\cdot(\rho\vec{V}W) - W\vec{V}\cdot\nabla\rho - W\nabla\cdot(\rho\vec{V}) \\
&= \nabla\cdot(\rho\vec{V}W) - W\vec{V}\cdot\nabla\rho + W\frac{\partial\rho}{\partial t},
\end{aligned} \tag{93}$$

where the $W\,\vec{V}\cdot\nabla\rho$ and $\partial\rho/\partial t$ terms may be omitted under the assumption that the primary is homogeneous and incompressible. In this approximation, the attenuation rate becomes simply

$$\dot{E} = -\int \nabla\cdot(\rho\vec{V}W)d^3x = -\int \rho W\vec{V}\cdot\vec{n}\,dA, \tag{94}$$

$\vec{n}$ being the outward normal to the surface of the primary, and dA being an element of the surface area. It is now clear that, under the said assertions, it is sufficient to take into account only the radial elevation rate, not the horizontal distortion. This way, formula (92), in application to a unit mass, will get simplified to

$$\dot{E} = \left(-\frac{\partial W}{\partial r}\right)\vec{V}\cdot\vec{n} = \left(-\frac{\partial W}{\partial r}\right)\frac{d\zeta}{dt}, \tag{95}$$

---

[23] We are considering flexure in the linear approximation. Thus at each frequency χ the appropriate energy loss over a cycle, $\Delta E_{\text{cycle}}(\chi)$, depends solely on the maximal energy stored at that same frequency, $E_{\text{peak}}(\chi)$.

ζ standing for the vertical displacement (which is, of course, delayed in time, compared to W). The amount of energy dissipated over a time interval (t_o, t) is then

$$\Delta E = \int_{t_o}^{t} \left(-\frac{\partial W}{\partial r}\right) d\zeta. \tag{96}$$

We shall consider the simple case of an equatorial moon on a circular orbit. At each point of the planet, the variable part of the tidal potential produced by this moon will read

$$W = W_o \, \cos \chi t, \tag{97}$$

the tidal frequency being given by

$$\chi = 2 \, |n - \omega_p|, \tag{98}$$

n being the secondary's mean motion, and ω_p being the primary's spin rate. Let g denote the surface free-fall acceleration. An element of the planet's surface lying beneath the satellite's trajectory will then experience a vertical elevation of

$$\zeta = h_2 \, \frac{W_o}{g} \, \cos(\chi t - |\epsilon|), \tag{99}$$

h_2 being the corresponding Love number[24], and $|\epsilon|$ being the *positive*[25] phase lag, which for the principal tidal frequency is simply the double geometric angle δ subtended at the primary's centre between the directions to the secondary and to the main bulge:

$$|\epsilon| = 2 \, \delta. \tag{100}$$

Accordingly, the vertical velocity of this element of the planet's surface will amount to

$$u = \dot{\zeta} = - \, h_2 \, \chi \, \frac{W_o}{g} \, \sin(\chi t - |\epsilon|) = - \, h_2 \, \chi \, \frac{W_o}{g} \, (\sin \chi t \, \cos |\epsilon| - \cos \chi t \, \sin |\epsilon|). \tag{101}$$

The expression for the velocity has such a simple form because in this case the instantaneous frequency χ is constant. The satellite generates two bulges—on the facing and opposite sides of the planet—so each point of the surface is uplifted twice through a cycle. This entails the factor of two in the expression (98) for the frequency. The phase in (100), too, is doubled, though the necessity of this is less evident,—see Appendix A1 to Efroimsky and Lainey (2007).

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[24] For a homogeneous incompressible body, $k_2 = (3/5)h_2$, for which reason (99) and the subsequent equations with h_2 can equally be written as proportional to k_2. The formulation employing k_2 is more fundamental, as it can, in principle, be generalised to a compressible body with a radially changing density. Indeed, whatever the properties of the primary body are, the dissipation rate in it is equal to the rate of change of the primary's spin energy plus the rate of change of the orbital energy in the system. Both the latter and the former are proportional to k_2.

[25] Were we not considering the simple case of a circular orbit, then, rigorously speaking, the expression for W would read not as $W_o \cos \chi t$ but as $W_o \cos \omega_{\text{tidal}} t$, the tidal frequency ω_{tidal} taking both positive and negative values, and the physical frequency of flexure being $\chi \equiv |\omega_{\text{tidal}}|$. Accordingly, the expression for ζ would contain not $\cos(\chi t - |\epsilon|)$ but $\cos(\omega_{\text{tidal}} t - \epsilon)$. As we saw in Eq. 24, the sign of ϵ is always the same as that of ω_{tidal}. For this reason, one may simply deal with the physical frequency $\chi \equiv |\omega_{\text{tidal}}|$ and with the absolute value of the phase lag, $|\epsilon|$.

The energy dissipated over a time cycle $T = 2\pi/\chi$, per unit mass, will, in neglect of horizontal displacements, be

$$
\Delta E_{\text{cycle}} = \int_0^T u \left(-\frac{\partial W}{\partial r} \right) dt = - \left(-h_2 \, \chi \, \frac{W_o}{g} \right) \frac{\partial W_o}{\partial r}
$$
$$
\times \int_{t=0}^{t=T} \cos \chi t \, (\sin \chi t \, \cos |\epsilon| - \cos \chi t \, \sin |\epsilon|) \, dt
$$
$$
= - h_2 \, \chi \, \frac{W_o}{g} \frac{\partial W_o}{\partial r} \, \sin |\epsilon| \, \frac{1}{\chi} \int_{\chi t=0}^{\chi t=2\pi} \cos^2 \chi t \, d(\chi t)
$$
$$
= - h_2 \, \frac{W_o}{g} \frac{\partial W_o}{\partial r} \, \pi \, \sin |\epsilon|, \tag{102}
$$

while the peak energy stored in the system during the cycle will read:

$$
E_{\text{peak}} = \int_{|\epsilon|/\chi}^{T/4} u \left(-\frac{\partial W}{\partial r} \right) dt = - \left(-h_2 \, \chi \, \frac{W_o}{g} \right) \frac{\partial W_o}{\partial r}
$$
$$
\times \int_{t=|\epsilon|/\chi}^{t=T/4} \cos \chi t \, (\sin \chi t \, \cos |\epsilon| - \cos \chi t \, \sin |\epsilon|) \, dt
$$
$$
= \chi \, h_2 \, \frac{W_o}{g} \frac{\partial W_o}{\partial r} \left[\frac{\cos |\epsilon|}{\chi} \int_{\chi t=|\epsilon|}^{\chi t=\pi/2} \cos \chi t \, \sin \chi t \, d(\chi t) - \frac{\sin |\epsilon|}{\chi} \right.
$$
$$
\left. \times \int_{\chi t=|\epsilon|}^{\chi t=\pi/2} \cos^2 \chi t \, d(\chi t) \right]. \tag{103}
$$

In the appropriate expression in Appendix A1 to Efroimsky and Lainey (2007), the lower limit of integration was erroneously set to be zero. To understand that in reality integration over χt should begin from $|\epsilon|$, one should superimpose the plots of the two functions involved, $\cos \chi t$ and $\sin(\chi t - |\epsilon|)$. The maximal energy gets stored in the system after integration through the entire interval over which both functions have the same sign. Hence $\chi t = |\epsilon|$ as the lower limit.

Evaluation of the integrals entails:

$$
E_{\text{peak}} = h_2 \, \frac{W_o}{g} \frac{\partial W_o}{\partial r} \left[\frac{1}{2} \, \cos |\epsilon| - \frac{1}{2} \left(\frac{\pi}{2} - |\epsilon| \right) \, \sin |\epsilon| \right] \tag{104}
$$

whence

$$
Q^{-1} = \frac{-\Delta E_{\text{cycle}}}{2\pi \, E_{\text{peak}}} = \frac{1}{2\pi} \frac{\pi \, \sin |\epsilon|}{\frac{1}{2} \cos |\epsilon| - \frac{1}{2} \left(\frac{\pi}{2} - |\epsilon| \right) \sin |\epsilon|}
$$
$$
= \frac{\tan |\epsilon|}{1 - \left(\frac{\pi}{2} - |\epsilon| \right) \tan |\epsilon|}. \tag{105}
$$

As can be seen from (105), both the product $\sin \epsilon \, \cos \epsilon$ and the appropriate component of the torque attain their maxima when $Q = 1 - \pi/4$.

Usually, $|\epsilon|$ is small, and we arrive at the customary expression

$$
Q^{-1} = \tan |\epsilon| + O(\epsilon^2). \tag{106}
$$

In the opposite situation, when $Q \to 0$ and $|\epsilon| \to \pi/2$, it is convenient to consider the small difference

$$\xi \equiv \frac{\pi}{2} - |\epsilon|, \tag{107}$$

in terms whereof the inverse quality factor will read:

$$Q^{-1} = \frac{\cot \xi}{1 - \xi \cot \xi} = \frac{1}{\tan \xi - \xi} = \frac{1}{z - \arctan z} = \frac{1}{\frac{1}{3} z^3 \left[1 - \frac{3}{5} z^2 + O(z^4)\right]}, \tag{108}$$

where $z \equiv \tan \xi$ and, accordingly, $\xi = \arctan z = z - \frac{1}{3} z^3 + \frac{1}{5} z^5 + O(z^7)$. The latter may, of course, be rewritten as

$$z^3 \left[1 - \frac{3}{5} z^2 + O(z^4)\right] = 3 Q \tag{109}$$

or, the same, as

$$z = (3 Q)^{1/3} \left[1 + \frac{1}{5} z^2 + O(z^4)\right]. \tag{110}$$

While the zeroth approximation is simply $z = (3Q)^{1/3} + O(Q)$, the first iteration gives:

$$\tan \xi \equiv z = (3Q)^{1/3} \left[1 + \frac{1}{5}(3Q)^{2/3} + O(Q^{4/3})\right] = q \left[1 + \frac{1}{5}q^2 + O(q^4)\right], \tag{111}$$

with $q = (3Q)^{1/3}$ playing the role of a small parameter.

We now see that the customary relation (106) should be substituted, for large lags, i.e., for small[26] values of Q, with:

$$\tan |\epsilon| = (3Q)^{-1/3} \left[1 - \frac{1}{5} (3Q)^{2/3} + O(Q^{4/3})\right] \tag{112}$$

The formulae for the tidal potential and torque contain a multiplier $\sin \epsilon \, \cos \epsilon$, whose absolute value can, for our purposes, be written down as

$$\sin |\epsilon| \cos |\epsilon| = \cos \xi \, \sin \xi = \frac{\tan \xi}{1 + \tan^2 \xi} = \frac{q \left[1 + \frac{1}{5}q^2 + O(q^4)\right]}{1 + q^2 \left[1 + O(q^2)\right]}$$

$$= (3Q)^{1/3} \left[1 - \frac{4}{5}(3Q)^{2/3} + O(Q^{4/3})\right], \tag{113}$$

whence

$$\sin \epsilon \, \cos \epsilon = \pm (3Q)^{1/3} \left[1 - \frac{4}{5}(3Q)^{2/3} + O(Q^{4/3})\right], \tag{114}$$

an expression vanishing for $Q \to 0$. Be mindful that both ϵ_{2200} and the appropriate component of the torque change their sign on the satellite crossing the synchronous orbit.

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[26] The afore-employed expansion of $\arctan z$ is valid for $|z| < 1$. This inequality, along with (108), entails: $Q = z - \arctan z < 1 - \pi/4$.

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